

DECOMPOSITION NUMBERS OF QUANTIZED WALLED BRAUER ALGEBRAS

HEBING RUI AND LINLIANG SONG

ABSTRACT. In this paper, we establish explicit relationship between decomposition numbers of quantized walled Brauer algebras and those for either Hecke algebras associated to certain symmetric groups or (rational) q -Schur algebras over a field κ . This enables us to use Ariki's result [2] and Varagnolo-Vasserot's result [35] to compute such decomposition numbers via inverse Kazhdan-Lusztig polynomials associated with affine Weyl groups of type A if the ground field is \mathbb{C} .

1. INTRODUCTION

The quantized walled Brauer algebra $\mathcal{B}_{r,s}$ with single parameter was introduced by Kosuda and Murakami [27] in order to study mixed tensor products of natural module and its dual over quantum general linear group $U_q(\mathfrak{gl}_n)$ over \mathbb{C} . In [28], Leduc introduced quantized walled Brauer algebras $\mathcal{B}_{r,s}$ with two parameters ρ and q . They are associative algebras over a commutative ring R containing 1.

It is proved in [21] that $\mathcal{B}_{r,s}$ is cellular over R in the sense of [22]. Using standard results on representations of cellular algebras in [22], we classified irreducible $\mathcal{B}_{r,s}$ -modules over an arbitrary field κ in [33]. Further, we gave a criterion on the semisimplicity of $\mathcal{B}_{r,s}$ over κ . A further question is to compute dimensions of irreducible $\mathcal{B}_{r,s}$ -modules in non semisimple case. This can be solved in theory by determining the multiplicity of any irreducible module in a cell (or standard) module of $\mathcal{B}_{r,s}$. Such a multiplicity is called a decomposition number of $\mathcal{B}_{r,s}$.

The aim of this paper is to compute decomposition numbers of $\mathcal{B}_{r,s}$ over \mathbb{C} . Recently, various authors have used a variety of techniques to determine decomposition numbers of Brauer-type algebras. Predominantly this has been via internal considerations [6, 7, 29, 36]. But the current paper is more in the spirit of Donkin-Tange [18], in that it relates these numbers to a Hecke or quantum group setting via Schur-Weyl duality.

By our result on the semisimplicity of quantized walled Brauer algebras in [33], we need to compute decomposition numbers of $\mathcal{B}_{r,s}$ under the assumptions either $\rho^2 \in q^{2\mathbb{Z}}$ or not. In the first case, we classify singular vectors of mixed tensor product of natural module and its dual over $U_q(\mathfrak{gl}_n)$ over κ . Via the explicit description on such singular vectors, we establish relationships between Weyl modules, partial tilting modules of rational q -Schur algebras and cell modules, principle indecomposable modules of

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$\mathcal{B}_{r,s}$. This proves that decomposition numbers of $\mathcal{B}_{r,s}$ can be determined via those for (rational) q -Schur algebras. In the second case, we use Schur functors in [33, §4] to set up relationship between decomposition numbers of $\mathcal{B}_{r,s}$ and those for Hecke algebras associated to symmetric groups. This enables us to use Ariki and Varagnolo-Vasserot's results [2, 35] to compute decomposition numbers of $\mathcal{B}_{r,s}$ via the values of inverse Kazhdan-Lusztig polynomials at $q = 1$ when the ground field is \mathbb{C} . As a by-product, we give some partial results on blocks of $\mathcal{B}_{r,s}$ over a field κ .

We organize our paper as follows. In section 2, we recall the definition of $\mathcal{B}_{r,s}$ and give some of its properties from [33]. In section 3, we establish explicit relationship between the decomposition numbers of $\mathcal{B}_{r,s}$ over κ and those for Hecke algebras associated to certain symmetric groups under the assumption that $\rho^2 \notin q^{2\mathbb{Z}}$. In section 4, we classify singular vectors of mixed tensor product of natural module and its dual over quantum general linear group $U_q(\mathfrak{gl}_n)$ over a field κ . Via such results, we set up relationship between decomposition numbers of $\mathcal{B}_{r,s}$ over κ with $\rho^2 \in q^{2\mathbb{Z}}$ and those for (rational) q -Schur algebras in section 5. When the ground field is \mathbb{C} , by using Ariki [2], Varagnolo-Vasserot's results [35] on the decomposition numbers for Hecke algebras and q -Schur algebras, we obtain the decomposition numbers of $\mathcal{B}_{r,s}$ no matter whether q is a root of unity or not. By the way, we will also give some partial results on blocks of $\mathcal{B}_{r,s}$ over κ .

2. THE QUANTIZED WALLED BRAUER ALGEBRA

In this section, we recall the definition of quantized walled Brauer algebras and state some of its properties from [33].

Let $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials in indeterminate q . The Hecke algebra \mathcal{H}_r associated to symmetric group \mathfrak{S}_r is an associative algebra over \mathcal{Z} , with generators g_1, g_2, \dots, g_{r-1} subject to the defining relations: $(g_i - q)(g_i + q^{-1}) = 0$, $1 \leq i \leq r-1$, and $g_i g_j = g_j g_i$, if $|i - j| > 1$, and $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, $1 \leq i < r-1$.

Let R be the localization of $\mathbb{Z}[q, q^{-1}, \rho, \rho^{-1}]$ at $q - q^{-1}$, and let $\delta = (\rho - \rho^{-1})(q - q^{-1})^{-1} \in R$. Fix $r, s \in \mathbb{Z}^{>0}$. The quantized walled Brauer algebra $\mathcal{B}_{r,s}$ [28] is the associative R -algebra with generators e_1, g_i, g_j^* , $1 \leq i \leq r-1$ and $1 \leq j \leq s-1$ such that g_i 's are generators of \mathcal{H}_r and g_j^* 's are generators of \mathcal{H}_s . Further, the following equalities hold if they make sense:

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|---|--|
| a) $g_i e_1 = e_1 g_i$, $g_i^* e_1 = e_1 g_i^*$, $i \neq 1$, | d) $g_i g_j^* = g_j^* g_i$, |
| b) $e_1 g_1 e_1 = \rho e_1 = e_1 g_1^* e_1$, | e) $e_1 g_1^{-1} g_1^* e_1 g_1 = e_1 g_1^{-1} g_1^* e_1 g_1^*$, |
| c) $e_1^2 = \delta e_1$, | f) $g_1 e_1 g_1^{-1} g_1^* e_1 = g_1^* e_1 g_1^{-1} g_1^* e_1$. |

Remark 2.1. In section 4, we will use Dipper-Doty-Stoll's presentation for $\mathcal{B}_{r,s}$ so as to use their result in [12, 13] to discuss singular vectors of mixed tensor products of quantum general linear groups. In that case, ρ and q in Dipper-Doty-Stoll's presentation is q^{-1} and ρ^{-1} in the current definition of $\mathcal{B}_{r,s}$, respectively.

Lemma 2.2. [21] *There is an R -linear anti-involution σ on $\mathcal{B}_{r,s}$ which fixes all generators e_1, g_i and g_j^* , $1 \leq i \leq r-1$ and $1 \leq j \leq s-1$.*

It is proved in [21] that $\mathcal{B}_{r,s}$ is cellular over R in the sense of [22]. In particular, the rank of $\mathcal{B}_{r,s}$ is $(r+s)!$. For any field κ which is an R -algebra, let $\mathcal{B}_{r,s,\kappa} = \mathcal{B}_{r,s} \otimes_R \kappa$.

Let e be the least positive integer such that $1 + q^2 + \cdots + q^{2(e-1)} = 0$ in κ . If there is no such a positive integer, i.e., $q^2 \in \kappa$ is not a root of unity, we set $e = \infty$. The following result has been proved by the authors in [33].

Theorem 2.3. [33, Theorem 6.10] *Suppose $r, s \in \mathbb{Z}^{>0}$. Then $\mathcal{B}_{r,s,\kappa}$ is (split) semisimple if and only if $e > \max\{r, s\}$ and one of the following conditions holds:*

- a) $\rho^2 \neq q^{2a}$ for any $a \in \mathbb{Z}$ with $|a| \leq r + s - 2$ if $\delta \neq 0$;
- b) $(r, s) \in \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ if $\delta = 0$.

When we classify singular vectors in mixed tensor products of natural module and its dual over quantum general linear groups, we will need explicit description of the cellular basis of $\mathcal{B}_{r,s}$ in [33, Theorem 3.7] as follows. We need some preparations before we state it.

A composition λ of n with at most d parts is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ such that $|\lambda| := \sum_{i=1}^d \lambda_i = n$. If $\lambda_i \geq \lambda_{i+1}$, $1 \leq i \leq d-1$, then λ is called a partition of n with at most d parts. Let $\Lambda(d, n)$ (resp. $\Lambda^+(d, n)$) be the set of all compositions (resp. partitions) of n with at most d parts. We also use $\Lambda^+(n)$ to denote the set of all partitions of n . It is known that $\Lambda^+(d, n)$ is a poset with dominant order \leq as a partial order on it. More explicitly, $\lambda \leq \mu$ for $\lambda, \mu \in \Lambda^+(d, n)$ if $\sum_{j=1}^i \lambda_j \leq \sum_{j=1}^i \mu_j$ for all possible $i \leq d$. Write $\lambda \triangleleft \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$.

Let $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda^+(n)$. The Young diagram $[\lambda]$ is a collection of boxes (or nodes) arranged in left-justified rows with λ_i boxes in the i -th row of $[\lambda]$. We use (i, j) to denote the box p if p is in i -th row and j -th column. A box (i, λ_i) (resp., $(i, \lambda_i + 1)$) is called a removable (resp., an addable) node of λ (or $[\lambda]$) if $\lambda_i - 1 \geq \lambda_{i+1}$ (resp. $\lambda_{i-1} \geq \lambda_i + 1$). Let $\mathcal{R}(\lambda)$ (resp., $\mathcal{A}(\lambda)$) be the set of all removable (resp., addable) boxes of λ . We use $\lambda \setminus p$ to denote the partition obtained from λ by removing the removable node p . Similarly, we use $\lambda \cup p$ to denote the partition obtained from λ by adding the addable node p .

A λ -tableau \mathfrak{s} is obtained by inserting elements i with $1 \leq i \leq d$ into $[\lambda]$. Let μ_i be the number of i appearing in \mathfrak{s} . Then $\mu = (\mu_1, \mu_2, \dots, \mu_d) \in \Lambda(d, n)$. In this case, \mathfrak{s} is called a λ -tableau of type μ . If the entries of \mathfrak{s} increase strictly down the columns and weakly increase along the rows, then \mathfrak{s} is called a semistandard λ -tableau of type μ . If we switch the role between columns and rows of \mathfrak{s} , then \mathfrak{s} is called a column semistandard λ -tableau of type μ . Let $\omega = (1, 1, \dots, 1) \in \Lambda^+(n)$. A λ -tableau \mathfrak{s} is said to be standard if and only if it is a semi-standard λ -tableau of type ω . Let $\mathcal{T}^{std}(\lambda)$ be the set of all standard λ -tableaux.

Now, we focus on λ -tableaux \mathfrak{s} of type ω . Such tableaux will be called λ -tableaux. The symmetric group \mathfrak{S}_n acts on \mathfrak{s} by permuting its entries. Let \mathfrak{t}^λ (resp. \mathfrak{t}_λ) be the λ -tableau obtained from $[\lambda]$ by adding $1, 2, \dots, n$ from left to right along the rows

(resp. from top to bottom along the columns). For example, if $\lambda = (4, 3, 1)$, then

$$\mathbf{t}^\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & & & \\ \hline \end{array}, \quad \text{and } \mathbf{t}_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 & \\ \hline 3 & & & \\ \hline \end{array}. \quad (2.1)$$

We write $w = d(\mathfrak{s})$ if $\mathbf{t}^\lambda w = \mathfrak{s}$. Then $d(\mathfrak{s})$ is uniquely determined by \mathfrak{s} .

Fix r and s and let

$$\Lambda_{r,s} = \{(f, \lambda) | \lambda \in \Lambda_{r,s}^f, 0 \leq f \leq \min\{r, s\}\}, \quad (2.2)$$

where $\Lambda_{r,s}^f = \Lambda^+(r-f) \times \Lambda^+(s-f)$. So, each $\lambda \in \Lambda_{r,s}^f$ is of form $(\lambda^{(1)}, \lambda^{(2)})$. We say that $(f, \lambda) \supseteq (\ell, \mu)$ if either $f > \ell$ or $f = \ell$ and $\lambda \supseteq \mu$ in the sense $\lambda^{(i)} \supseteq \mu^{(i)}$, $i = 1, 2$. We write $(f, \lambda) \supset (\ell, \mu)$ if $(f, \lambda) \supseteq (\ell, \mu)$ and $(f, \lambda) \neq (\ell, \mu)$. Then $\Lambda_{r,s}$ is a poset.

Given a $\lambda \in \Lambda_{r,s}^f$, we define $\mathbf{t}^\lambda = (\mathbf{t}^{\lambda^{(1)}}, \mathbf{t}^{\lambda^{(2)}})$ where $\mathbf{t}^{\lambda^{(1)}}$ and $\mathbf{t}^{\lambda^{(2)}}$ are defined similarly as (2.1). The only difference is that we have to use $f+i$ instead of i in (2.1). Similarly, we have \mathbf{t}_λ .

Example 2.4. Suppose $(r, s) = (2, 7)$, $f = 1$ and $(\lambda^{(1)}, \lambda^{(2)}) = ((1), (3, 2, 1))$. We have

$$\mathbf{t}^\lambda = \left(\begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 5 & 6 & \\ \hline 7 & & \\ \hline \end{array} \right) \quad \text{and} \quad \mathbf{t}_\lambda = \left(\begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 5 & 7 \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline \end{array} \right). \quad (2.3)$$

For each $\lambda \in \Lambda_{r,s}^f$, let $\mathcal{T}^{std}(\lambda^{(i)})$ be the set of standard $\lambda^{(i)}$ -tableaux which are obtained from usual standard tableaux by using $f+j$ instead of j . Let $\mathcal{T}^{std}(\lambda) = \mathcal{T}^{std}(\lambda^{(1)}) \times \mathcal{T}^{std}(\lambda^{(2)})$.

For each partition λ of n , let \mathfrak{S}_λ be the Young subgroup of \mathfrak{S}_n with respect to λ . Let $\mathbf{n}_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-\ell(w)} g_w$ and let $\mathbf{m}_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{\ell(w)} g_w$. Then $\mathbf{n}_\lambda g_i = -q^{-1} \mathbf{n}_\lambda$ and $\mathbf{m}_\lambda g_i = q \mathbf{m}_\lambda$, if $s_i \in \mathfrak{S}_\lambda$.

Recall that σ is the anti-involution on $\mathcal{B}_{r,s}$ given in Lemma 2.2. If $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{std}(\lambda)$ with $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$ and $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2)$, we define

$$\mathbf{n}_{\mathfrak{s}\mathfrak{t}} = \sigma(g_{d(\mathfrak{s})}) \mathbf{n}_\lambda g_{d(\mathfrak{t})}, \quad (2.4)$$

where $\mathbf{n}_\lambda = \mathbf{n}_{\lambda^{(1)}} \mathbf{n}_{\lambda^{(2)}}$, $g_{d(\mathfrak{s})} = g_{d(\mathfrak{s}_1)} g_{d(\mathfrak{s}_2)}^*$, $g_{d(\mathfrak{t})} = g_{d(\mathfrak{t}_1)} g_{d(\mathfrak{t}_2)}^*$, $d(\mathfrak{s}) = d(\mathfrak{s}_1) d(\mathfrak{s}_2)$, and $d(\mathfrak{t}) = d(\mathfrak{t}_1) d(\mathfrak{t}_2)$. We remark that we use s_1, \dots, s_{r-1} and s_1^*, \dots, s_{s-1}^* to denote generators of \mathfrak{S}_r and \mathfrak{S}_s , respectively.

Fix $r, s \in \mathbb{Z}^{>0}$ and $f \in \mathbb{N}$ with $f \leq \min\{r, s\}$. Let

$$\mathcal{D}_{r,s}^f = \{s_{f,i_f} s_{f,j_f}^* \cdots s_{1,i_1} s_{1,j_1}^* \mid k \leq j_k, 1 \leq i_1 < i_2 < \cdots < i_f \leq r\}.$$

For each $(f, \lambda) \in \Lambda_{r,s}$, we define $I(f, \lambda) = \mathcal{T}^{std}(\lambda) \times \mathcal{D}_{r,s}^f$.

In [33], we defined $e_{i,j} = g_{1,i}^{-1} g_{j,1}^* e_1 g_{1,i} (g_{j,1}^*)^{-1}$, for all $1 \leq i \leq r$ and $1 \leq j \leq s$. If $i = j$, we denote $e_{i,j}$ by e_i . For any positive integer $f \leq \min\{r, s\}$, let $e^f = e_1 e_2 \cdots e_f$. If $f = 0$, we denote e^f by 1.

For any $(\mathfrak{s}, e), (\mathfrak{t}, d) \in I(f, \lambda)$, we define

$$C_{(\mathfrak{s}, e)(\mathfrak{t}, d)} = \sigma(g_e) e^f \mathbf{n}_{\mathfrak{s}\mathfrak{t}} g_d. \quad (2.5)$$

The following result, which has been proved in [33, Theorem 3.7] can also be obtained from [21, Theorem 6.13]¹

Theorem 2.5. *Let $\mathcal{B}_{r,s}$ be the quantized walled Brauer algebra over R . Then \mathcal{C} is a cellular R -basis of $\mathcal{B}_{r,s}$ over the poset $\Lambda_{r,s}$ in the sense of [22], where*

$$\mathcal{C} = \cup_{(f,\lambda) \in \Lambda_{r,s}} \{C_{(\mathfrak{s},e)(\mathfrak{t},d)} \mid (\mathfrak{s},e), (\mathfrak{t},d) \in I(f,\lambda)\}.$$

The required anti-involution σ is the one given in Lemma 2.2.

Recall that κ is a field which is an R -algebra and $\mathcal{B}_{r,s,\kappa} = \mathcal{B}_{r,s} \otimes_R \kappa$. In this paper, we consider right $\mathcal{B}_{r,s,\kappa}$ -modules. By standard results on the representations of cellular algebras in [22], we have the right cell module $C(f,\lambda)$ for each $(f,\lambda) \in \Lambda_{r,s}$, which is spanned by $\{e^f \mathbf{n}_{\mathfrak{t}\lambda} g_d + \mathcal{B}_{r,s,\kappa}^{\triangleright(f,\lambda)} \mid (\mathfrak{s},d) \in \mathcal{T}^{std}(\lambda) \times \mathcal{D}_{r,s}^f\}$ as κ -space, where $\mathcal{B}_{r,s,\kappa}^{\triangleright(f,\lambda)}$ is a subspace of $\mathcal{B}_{r,s,\kappa}$ spanned by $\cup_{(\ell,\mu) \in \Lambda_{r,s}} \{C_{(\mathfrak{s},e)(\mathfrak{t},d)} \mid (\mathfrak{s},e), (\mathfrak{t},d) \in I(\ell,\mu)\}$ with $(f,\lambda) \triangleleft (\ell,\mu)$. In fact, $\mathcal{B}_{r,s,\kappa}^{\triangleright(f,\lambda)}$ is a two-sided ideal of $\mathcal{B}_{r,s,\kappa}$.

For bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, let $\lambda' = (\mu^{(1)}, \mu^{(2)})$ where $\mu^{(i)}$ is the conjugate of $\lambda^{(i)}$ for $i = 1, 2$. We call λ' the conjugate of λ . We set $\mathbf{m}_\lambda = \mathbf{m}_{\lambda^{(1)}} \mathbf{m}_{\lambda^{(2)}}$. Note that the current $\mathbf{m}_{\lambda^{(i)}}$ is obtained from usual one by using g_{f+j} (resp. g_{f+j}^*) instead of g_j (resp. g_j^*) if $i = 1$ (resp. $i = 2$). The following result, which will be needed in section 4, has been proved in [33].

Proposition 2.6. *For each $(f,\lambda) \in \Lambda_{r,s}$, let $\tilde{C}(f,\lambda) := e^f \mathbf{m}_{\lambda'} g_d(\mathfrak{t}_{\lambda'}) \mathbf{n}_\lambda \mathcal{B}_{r,s} \pmod{\mathcal{B}_{r,s}^{f+1}}$, where $\mathcal{B}_{r,s}^{f+1}$ is the two-sided ideal of $\mathcal{B}_{r,s}$ generated by e^{f+1} . As right $\mathcal{B}_{r,s}$ -modules, $C(f,\lambda) \cong \tilde{C}(f,\lambda)$.*

It follows from standard results on the representations of cellular algebras in [22] that there is an invariant form, say $\phi_{f,\lambda}$, on each cell module $C(f,\lambda)$. Let $D^{f,\lambda} = C(f,\lambda) / \text{Rad } \phi_{f,\lambda}$, where $\text{Rad } \phi_{f,\lambda}$ is the radical of $\phi_{f,\lambda}$. Then $D^{f,\lambda}$ is either zero or absolutely irreducible, and all non-zero $D^{f,\lambda}$'s form a complete set of all non-isomorphic irreducible $\mathcal{B}_{r,s,\kappa}$ -modules.

Recall that a partition λ is called e -restricted if $\lambda_i - \lambda_{i+1} < e$ for all $i \geq 1$. If $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, then λ is e -restricted if and only if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e -restricted. If λ' is e -restricted, then λ is called e -regular. In [33], we have proved that $D^{f,\lambda} \neq 0$ if and only if λ is e -restricted provided that one of the conditions holds: (a) $\delta \neq 0$, (b) $\delta = 0$ and $r \neq s$, (c) $\delta = 0$, $r = s$ and $f \neq r$. This enables us to prove the following result in [33].

Theorem 2.7. [33, Theorem 5.3] *Let $\mathcal{B}_{r,s,\kappa}$ be the quantized walled Brauer algebra over the field κ .*

- a) *If either $\delta \neq 0$ or $\delta = 0$ and $r \neq s$, then the non-isomorphic irreducible $\mathcal{B}_{r,s,\kappa}$ -modules are indexed by $\{(f,\lambda) \mid 0 \leq f \leq \min\{r,s\}, \lambda \text{ being } e\text{-restricted}\}$.*
- b) *If $\delta = 0$ and $r = s$, then the non-isomorphic irreducible $\mathcal{B}_{r,s,\kappa}$ -modules are indexed by $\{(f,\lambda) \mid 0 \leq f < r, \lambda \text{ being } e\text{-restricted}\}$.*

¹Enyang [21] has proved that any cellular basis of Hecke algebras can be lifted to get a cellular basis of $\mathcal{B}_{r,s}$.

We denote by $[C(f, \lambda) : D^{\ell, \mu}]$ the multiplicity of $D^{\ell, \mu}$ in $C(f, \lambda)$. Such a non-negative integer will be called a decomposition number of $\mathcal{B}_{r, s, \kappa}$.

In the remaining part of this paper, we establish the explicit relationship between decomposition numbers of $\mathcal{B}_{r, s}$ and those for Hecke algebras and q -Schur algebras. Using Ariki, Varagnolo-Vasserot's results on decomposition numbers of Hecke algebras and q -Schur algebras in [2, 35] yields the formulae on the decomposition numbers of $\mathcal{B}_{r, s}$, as required if the ground field is \mathbb{C} .

3. DECOMPOSITION NUMBERS OF $\mathcal{B}_{r, s, \kappa}$ WITH $\rho^2 \notin q^{2\mathbb{Z}}$

In this section, we consider $\mathcal{B}_{r, s, \kappa}$ over κ such that $\rho^2 \neq q^{2a}$ for all $a \in \mathbb{Z}$ with $|a| \leq r + s - 2$. So, $\rho^2 \neq 1$ and $\delta \neq 0$. By Theorem 2.3, $\mathcal{B}_{r, s, \kappa}$ is semisimple if $e > \max\{r, s\}$. So, we consider $\mathcal{B}_{r, s, \kappa}$ under the assumption $e \leq \max\{r, s\}$. In this case, we will prove that decomposition numbers of $\mathcal{B}_{r, s, \kappa}$ are determined by those for Hecke algebras associated with certain symmetric groups. We remark that blocks of $\mathcal{B}_{r, s, \kappa}$ will also be classified.

Let $\mathcal{B}_{r, s, \kappa}\text{-mod}$ be the category of right $\mathcal{B}_{r, s, \kappa}$ -modules. In [33, §4], we define the exact functor $\mathcal{F}_{r, s} : \mathcal{B}_{r, s, \kappa}\text{-mod} \rightarrow \mathcal{B}_{r-1, s-1, \kappa}\text{-mod}$ and right exact functor $\mathcal{G}_{r, s} : \mathcal{B}_{r, s, \kappa}\text{-mod} \rightarrow \mathcal{B}_{r+1, s+1, \kappa}\text{-mod}$ ². We call $\mathcal{F}_{r, s}$ the Schur functor. By abuse of notations, we use \mathcal{F} and \mathcal{G} instead of $\mathcal{F}_{r, s}$ and $\mathcal{G}_{r, s}$, respectively. We remark that we consider right cell modules of $\mathcal{B}_{r, s, \kappa}$ in this section.

Lemma 3.1. *Let $(\ell, \mu), (f, \lambda) \in \Lambda_{r, s}$ with μ being e -restricted. Then $[C(f, \lambda) : D^{\ell, \mu}] \neq 0$ only if $\ell = f$.*

Proof. We prove our result by induction on $r + s$. Since we are assuming $r, s \in \mathbb{Z}^{>0}$, we have $r + s \geq 2$. It is not difficult to check the result for $r + s = 2$. In this case, $r = s = 1$, $f = 1$ and $\lambda = (\emptyset, \emptyset)$ if $f \neq 0$.

In general, we can assume $r \geq 2$. Suppose $\ell > 0$. We apply the exact functor \mathcal{F} to both $D^{\ell, \mu}$ and $C(f, \lambda)$. By [33, Lemma 4.3], we have $\mathcal{F}(C(f, \lambda)) \cong C(f - 1, \lambda)$ for left cell modules. In fact, this holds for right cell modules. Since we are assuming $[C(f, \lambda) : D^{\ell, \mu}] \neq 0$, we have $f \geq \ell \geq 1$. By [23, 6.2g], $\mathcal{F}(D^{\ell, \mu})$ is either zero or a simple $\mathcal{B}_{r-1, s-1, \kappa}$ -module and each simple $\mathcal{B}_{r-1, s-1, \kappa}$ -module is of form $\mathcal{F}(D^{\ell, \mu})$ for some simple $\mathcal{B}_{r, s, \kappa}$ -module $D^{\ell, \mu}$. Mimicking arguments in the proof of [32, Lemma 2.9], we see that there is a non-trivial homomorphism from $C(\ell - 1, \mu)$ to $\mathcal{F}(D^{\ell, \mu})$, forcing $\mathcal{F}(D^{\ell, \mu}) \neq 0$. By Theorem 2.7, $D^{\ell-1, \mu} \neq 0$ and $\mathcal{F}(D^{\ell, \mu}) = D^{\ell-1, \mu}$, if $\ell \geq 1$. By the exactness of \mathcal{F} , we have

$$[C(f - 1, \lambda) : D^{\ell-1, \mu}] = [C(f, \lambda) : D^{\ell, \mu}] \neq 0. \quad (3.1)$$

Using induction assumption yields $f = \ell$.

Now, we assume $\ell = 0$. If $f = 0$, there is nothing to be proved. So, we assume $f \geq 1$. Let $\text{Res}^L M$ be the restriction of $\mathcal{B}_{r, s, \kappa}$ -module M to $\mathcal{B}_{r-1, s, \kappa}$. Note that

²In [33], we considered two functors $\mathcal{F}_{r, s}$ and $\mathcal{G}_{r, s}$ for left modules. However, one can prove similar results for right modules.

$D^{0,\mu}$ can be considered as $\mathcal{H}_r \otimes \mathcal{H}_s$ -module. So, $\text{Res}^L D^{0,\mu}$ can be considered as $\mathcal{H}_{r-1} \otimes \mathcal{H}_s$ -module. By modular branching rule for \mathcal{H}_r in [4], we find a removable node p of $\mu^{(1)}$ such that $D^{0,(\mu^{(1)} \setminus p, \mu^{(2)})}$ is in the socle of $\text{Res}^L D^{0,\mu}$. It is a composition factor of $\text{Res}^L C(f, \lambda)$. By [33, Theorem 4.15] and induction assumption, $f = 1$ and

$$[C(0, (\lambda^{(1)}, \lambda^{(2)} \cup p_2)) : D^{0,(\mu^{(1)} \setminus p, \mu^{(2)})}] \neq 0$$

for some $p_2 \in \mathcal{A}(\lambda^{(2)})$. Applying [33, Lemma 6.3] to $C(0, \mu)$, $C(0, (\mu^{(1)} \setminus p, \mu^{(2)}))$, $C(1, \lambda)$ and $C(0, (\lambda^{(1)}, \lambda^{(2)} \cup p_2))$ yields $\rho^2 = q^{2k}$ with $|k| = |\text{res}(p) + \text{res}(p_2)| \leq r + s - 2$, where $\text{res}(p) = j - i$ if p is in i -th row and j -th column. This is a contradiction. \square

Graham and Lehrer [22] defined a cell block of a cellular algebra, which is a equivalent class generated by the notion of cell linked. In our case, (f, λ) and (ℓ, μ) are said to be cell linked if either $D^{f,\lambda}$ is a composition factor of $C(\ell, \mu)$ or $D^{\ell,\mu}$ is a composition factor of $C(f, \lambda)$. By [22, 3.9.8], a block of irreducible modules for $\mathcal{B}_{r,s,\kappa}$ is the intersection of $\overline{\Lambda}_{r,s}$ with a cell block, where $\overline{\Lambda}_{r,s}$ consists of all $(f, \lambda) \in \Lambda_{r,s}$ with $D^{f,\lambda} \neq 0$. See Theorem 2.7 for the explicit description on $\overline{\Lambda}_{r,s}$.

Theorem 3.2. *Suppose $(f, \lambda), (\ell, \mu) \in \Lambda_{r,s}$.*

- a) *$C(f, \lambda)$ and $C(\ell, \mu)$ are in the same block if and only if $f = \ell$ and $C(0, \lambda)$ and $C(0, \mu)$ are in the same block.*
- b) *$[C(f, \lambda) : D^{\ell,\mu}] = \delta_{f,\ell} [C(0, \lambda) : D^{0,\mu}]$ for any μ being e -restricted.*

Proof. We remark that (b) follows from Lemma 3.1 and (3.1) immediately. We prove (a) as follows.

Without loss of any generality, we can assume that $C(f, \lambda)$ has the simple head $D^{f,\lambda}$ which is a composition factor of $C(\ell, \mu)$. By Lemma 3.1, $f = \ell$. Applying exact functor \mathcal{F} to both $D^{f,\lambda}$ and $C(\ell, \mu)$ repeatedly, we have that $D^{0,\lambda}$ is a composition factor of $C(0, \mu)$.

Conversely, let $D^{0,\lambda}$ be a composition factor of $C(0, \mu)$. Then, there are two submodules M_1, M_2 of $C(0, \mu)$ such that $D^{0,\lambda} \cong M_1/M_2$. By the right exactness of \mathcal{G} , there is an epimorphism from $\mathcal{G}(M_1)$ to $G(D^{0,\lambda})$. Similarly, we have an epimorphism from $\mathcal{G}(C(0, \lambda))$ to $G(D^{0,\lambda})$. By [33, Lemma 4.3a], $D^{0,\lambda} = \mathcal{F}\mathcal{G}(D^{0,\lambda})$, forcing $G(D^{0,\lambda}) \neq 0$. Since $C(0, \lambda)$ has the simple head $D^{0,\lambda}$, λ is e -restricted. By Theorem 2.7, $C(1, \lambda)$ has the simple $D^{1,\lambda}$, forcing an epimorphism from $G(D^{0,\lambda})$ to $D^{1,\lambda}$. So, $D^{1,\lambda}$ is a composition factor of $\mathcal{G}(M_1) \subset C(1, \mu)$. Using the previous arguments repeatedly, we have that $D^{f,\lambda}$ is a composition factor of $C(f, \mu)$. So, $C(f, \lambda)$ and $C(f, \mu)$ are in the same block. \square

By Theorem 3.2 and explicit description on blocks of Hecke algebras associated with symmetric groups in, e.g. [30], we know explicitly the description of blocks of non-semisimple $\mathcal{B}_{r,s,\kappa}$ under the assumption $\rho^2 \notin q^{2\mathbb{Z}}$. Further, since $C(0, \lambda)$ can be considered as the cell module of $\mathcal{H}_{r-f} \otimes \mathcal{H}_{s-f}$, $[C(f, \lambda) : D^{\ell,\mu}]$ can be computed by

Ariki's result [2] on the decomposition numbers of Hecke algebra associated to symmetric groups if the ground field is \mathbb{C} . More explicitly, such decomposition numbers are computed via inverse Kazhdan-Lusztig polynomials associated to certain affine Weyl groups of type A .

4. SINGULAR VECTORS OF THE MIXED TENSOR PRODUCT

Throughout, let $\mathbb{Q}(q)$ be the quotient field of \mathcal{Z} , where $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ is the ring of Laurent polynomials in indeterminate q .

Let P^\vee be the free \mathbb{Z} -module with basis h_1, \dots, h_n , and let $P^{\vee*}$ be its dual. Then $P^{\vee*}$ has a dual basis $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon_i(h_j) = \delta_{i,j}$, $1 \leq i, j \leq n$. The quantum general linear group $\mathbf{U}_q(\mathfrak{gl}_n)$ is an associative $\mathbb{Q}(q)$ -algebra generated by E_i, F_i , $1 \leq i \leq n-1$ and q^h , $h \in P^\vee$ subject to the defining relations

- a) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$, for any $h, h' \in P^\vee$,
- b) $q^h E_i q^{-h} = q^{\alpha_i(h)} E_i$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$,
- c) $q^h F_i q^{-h} = q^{-\alpha_i(h)} F_i$,
- d) $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}$ and $K_i = q^{h_i - h_{i+1}}$,
- e) $E_i E_j = E_j E_i$, for $|i - j| > 1$,
- f) $F_i F_j = F_j F_i$, for $|i - j| > 1$.
- g) $E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ for $|i - j| = 1$,
- h) $F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ for $|i - j| = 1$,

It is well known that $\mathbf{U}_q(\mathfrak{gl}_n)$ is a Hopf algebra such that the comultiplication Δ , counit ε and antipode S satisfy the following conditions:

- a) $\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i$,
- b) $\Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i$,
- c) $\Delta(q^h) = q^h \otimes q^h$,
- d) $S(F_i) = -K_i^{-1} F_i$, $S(E_i) = -E_i K_i$ and $S(q^h) = q^{-h}$,
- e) $\varepsilon(E_i) = \varepsilon(F_i) = 0$ and $\varepsilon(q^h) = 1$.

If we use q^{-1} instead of q , then the previous $\mathbf{U}_q(\mathfrak{gl}_n)$ is the quantum general linear group in [26] and the current E_i and F_j correspond to F_i and E_j in [26].

It is known that $\mathbf{U}_q(\mathfrak{gl}_n)$ has a \mathcal{Z} -Hopf-subalgebra $\mathbf{U}_{\mathcal{Z}}$, which is generated by q^h , and divided powers $E_i^{(\ell)} = \frac{E_i^\ell}{[\ell]!}$ and $F_i^{(\ell)} = \frac{F_i^\ell}{[\ell]!}$, for all $h \in P^\vee$ and all $\ell \in \mathbb{Z}^{>0}$, where $[\ell]! = [\ell][\ell-1] \cdots [1]$, $[\ell] = \frac{q^\ell - q^{-\ell}}{q - q^{-1}}$.

For each left $\mathbf{U}_{\mathcal{Z}}$ -module M , and $\lambda \in \mathbb{Z}^n$, define

$$M_\lambda = \{m \in M \mid q^{h_i} \cdot m = q^{\lambda_i} m, 1 \leq i \leq n\}.$$

Then λ is called a weight of M if $M_\lambda \neq 0$. In this case, M_λ is called the weight space of q^h acting on M . Further, each weight space of M is of form M_λ . If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then λ is called a **dominant weight**. Let $\mathfrak{X}^+(n)$ be the set of all $\lambda \in \mathbb{Z}^n$ with $\lambda_i \geq \lambda_{i+1}$ for all i , $1 \leq i \leq n-1$.

Lemma 4.1. [12] *Let V be the free \mathcal{Z} -module V with basis $\{v_1, v_2, \dots, v_n\}$. Let $V^* = \text{Hom}_{\mathcal{Z}}(V, \mathcal{Z})$ be the dual of V with dual basis $\{v_1^*, v_2^*, \dots, v_n^*\}$. Then both V and V^* are left $\mathbf{U}_{\mathcal{Z}}$ -module such that*

- a) $q^h v_j = q^{\varepsilon_j(h)} v_j$, $E_i v_j = \delta_{j,i+1} v_i$, $F_i v_j = \delta_{i,j} v_{i+1}$,
 - b) $q^h v_j^* = q^{-\varepsilon_j(h)} v_j^*$, $E_i v_j^* = -\delta_{i,j} q^{-1} v_{i+1}^*$, $F_i v_j^* = -\delta_{i+1,j} q v_i^*$,
- if all of them make sense.*

Proof. (a) has been given in [12] and (b) can be verified easily by using (a) and antipode S for $\mathbf{U}_q(\mathfrak{gl}_n)$. \square

In the remaining part of this paper, we denote V^* by W . Fix two positive integers r and s . Then the mixed tensor space $V^{r,s} := V^{\otimes r} \otimes W^{\otimes s}$, which was studied in [12, 13], is a left $\mathbf{U}_{\mathcal{Z}}$ -module. Given positive integers n, r, s , let

$$\begin{aligned} I(n, r) &= \{\mathbf{i} \mid \mathbf{i} = (i_r, i_{r-1}, \dots, i_1), 1 \leq i_j \leq n, 1 \leq j \leq r\}, \\ I^*(n, s) &= \{\mathbf{j} \mid \mathbf{j} = (i_1, i_2, \dots, i_s), 1 \leq i_j \leq n, 1 \leq j \leq s\}. \end{aligned} \quad (4.1)$$

Then the symmetric group $\mathfrak{S}_r \times \mathfrak{S}_s$ acts on the right of $I(n, r) \times I^*(n, s)$ by place permutation in the sense $(\mathbf{i}, \mathbf{j})w w^* = (\mathbf{i}w, \mathbf{j}w^*)$ for any $(\mathbf{i}, \mathbf{j}) \in I(n, r) \times I^*(n, s)$ and $w \in \mathfrak{S}_r$, and $w^* \in \mathfrak{S}_s$.

For each $(\mathbf{i}, \mathbf{j}) \in I(n, r) \times I^*(n, s)$, define

$$\lambda_k = \#\{\ell \mid i_\ell = k\} - \#\{\ell \mid j_\ell = k\},$$

for $1 \leq k \leq n$ and write $\text{wt}(\mathbf{i}, \mathbf{j}) = (\lambda_1, \dots, \lambda_n)$. We call $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, the weight of (\mathbf{i}, \mathbf{j}) . It is easy to see that (\mathbf{i}, \mathbf{j}) and (\mathbf{k}, \mathbf{l}) have the same weight if they are in the same $\mathfrak{S}_r \times \mathfrak{S}_s$ -orbit. However, the converse is not true.

For each $(\mathbf{i}, \mathbf{j}) \in I(n, r) \times I^*(n, s)$, define $v_{\mathbf{i}|\mathbf{j}} = v_{\mathbf{i}} \otimes v_{\mathbf{j}}^*$, where

$$v_{\mathbf{i}} = v_{i_r} \otimes v_{i_{r-1}} \otimes \dots \otimes v_{i_1}, \quad \text{and} \quad v_{\mathbf{j}}^* = v_{j_1}^* \otimes v_{j_2}^* \otimes \dots \otimes v_{j_s}^*. \quad (4.2)$$

Then $\{v_{\mathbf{i}|\mathbf{j}} \mid (\mathbf{i}, \mathbf{j}) \in I(n, r) \times I^*(n, s)\}$ is a \mathcal{Z} -basis of $V^{r,s}$. Obviously, the weight of $v_{\mathbf{i}|\mathbf{j}} \in V^{r,s}$ is the same as the weight of $(\mathbf{i}, \mathbf{j}) \in I(n, r) \times I^*(n, s)$.

Lemma 4.2. *Let $\Lambda(r, s) = \{\lambda \in \mathbb{Z}^n \mid \sum_{\lambda_i > 0} \lambda_i = r - f, \sum_{\lambda_j < 0} \lambda_j = f - s, 0 \leq f \leq \min\{r, s\}\}$. Then $\Lambda(r, s)$ is the set of weights of $V^{r,s}$.*

Proof. Easy exercise. \square

Lemma 4.3. *Given $r, s, n \in \mathbb{Z}^{>0}$ with $n \geq r + s$, let $\Lambda^+(r, s) = \Lambda(r, s) \cap \mathfrak{X}^+(n)$.*

- a) *There is a bijection $\phi : \Lambda_{r,s} \rightarrow \Lambda^+(r, s)$;*
- b) *If $\Lambda = \{\lambda + s\omega \mid \lambda \in \Lambda^+(r, s)\}$ with $\omega = (1, 1, \dots, 1) \in \mathbb{Z}^n$, then Λ is an ideal of $\Lambda^+(n, r + (n - 1)s)$ in the sense that $\lambda \in \Lambda$ if $\lambda \trianglelefteq \mu$ for some $\mu \in \Lambda$.*

Proof. Suppose $(f, \lambda) \in \Lambda_{r,s}$ such that $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, $l(\lambda^{(1)}) = k$ and $l(\lambda^{(2)}) = \ell$, where $l(\lambda^{(i)})$ is the maximal index j such that $\lambda_j^{(i)} \neq 0$. Since $n \geq r + s$, the required bijection ϕ sends (f, λ) to $\phi(f, \lambda)$ where

$$\phi(f, \lambda) = \underbrace{(\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_k^{(1)}, 0, \dots, 0, -\lambda_\ell^{(2)}, \dots, -\lambda_1^{(2)})}_n.$$

This proves (a). One can verify (b) by straightforward computation. \square

Let κ be a field which is a \mathcal{Z} -algebra, let $\mathbf{U}_\kappa = \mathbf{U}_{\mathcal{Z}} \otimes_{\mathcal{Z}} \kappa$. By abuse of notations, we denote $E_i^{(\ell)}$ (resp., $F_i^{(\ell)}$) by $E_i^{(\ell)} \otimes 1_\kappa$ (resp., $F_i^{(\ell)} \otimes 1_\kappa$).

Suppose M is a finite dimensional left \mathbf{U}_κ -module. If $0 \neq v \in M_\lambda$ for some $\lambda \in \mathfrak{X}^+(n)$, such that $E_i^\ell / [\ell]! v = 0$, $\forall i, \ell, 1 \leq i \leq n-1$ and $\ell > 0$, then v is called a **highest weight** (or **singular**) vector of M with highest weight λ .

In the remaining part of this section, we want to classify singular vectors of $V_\kappa^{r,s} = V^{\otimes r} \otimes W^{\otimes s} \otimes \kappa \cong V_\kappa^{\otimes r} \otimes W_\kappa^{\otimes s}$ over κ , provided that $n \geq r+s$. Since we are going to use Dipper-Doty-Stoll's results in [12, 13], we consider their presentation of $\mathcal{B}_{r,s}$ with $\rho = q^n$ over \mathcal{Z} . As mentioned before, Dipper-Doty-Stoll's presentation for $\mathcal{B}_{r,s}$ can be obtained from that in section 2 by using q^{-1} and ρ^{-1} instead of q, ρ , respectively. In this case, we still have $\rho^{-1} = (q^{-1})^n$.

Proposition 4.4. [12] *Let $\mathcal{B}_{r,s}$ be the quantized walled Brauer algebra over \mathcal{Z} with defining parameter $\rho = q^n$. Then $V^{r,s}$ is a right $\mathcal{B}_{r,s}$ -module over \mathcal{Z} such that, for any $(\mathbf{i}, \mathbf{j}) \in I(n, r) \times I^*(n, s)$,*

- a) $v_{\mathbf{i}|\mathbf{j}}e_1 = \delta_{i_1, j_1} q^{-n-1+2i_1} \sum_{s=1}^n v_{\hat{\mathbf{i}}} \otimes v_s \otimes v_s^* \otimes v_{\hat{\mathbf{j}}}^*$,
- b) $v_{\mathbf{i}|\mathbf{j}}g_k = q^{-1}v_{\mathbf{i}|\mathbf{j}}$, (resp., $v_{\mathbf{i}|\mathbf{j}}g_k^* = q^{-1}v_{\mathbf{i}|\mathbf{j}}$), if $i_k = i_{k+1}$, (resp., $j_k = j_{k+1}$),
- c) $v_{\mathbf{i}|\mathbf{j}}g_k = v_{\mathbf{i}s_k|\mathbf{j}}$, (resp., $v_{\mathbf{i}|\mathbf{j}}g_k^* = v_{\mathbf{i}|\mathbf{j}s_k^*}$) if $i_k < i_{k+1}$, (resp., $j_k > j_{k+1}$),
- d) $v_{\mathbf{i}|\mathbf{j}}g_k = v_{\mathbf{i}s_k|\mathbf{j}} + (q^{-1} - q)v_{\mathbf{i}|\mathbf{j}}$, (resp. $v_{\mathbf{i}|\mathbf{j}}g_k^* = v_{\mathbf{i}|\mathbf{j}s_k^*} + (q^{-1} - q)v_{\mathbf{i}|\mathbf{j}}$), if $i_k > i_{k+1}$, (resp., $j_k < j_{k+1}$),

where $\hat{\mathbf{i}}$ (resp., $\hat{\mathbf{j}}$) is obtained from \mathbf{i} (resp., \mathbf{j}) by dropping i_1 (resp., j_1).

Remark 4.5. In this paper, $v_{\mathbf{i}} = v_{i_r} \otimes \cdots \otimes v_{i_1}$ for $\mathbf{i} \in I(n, r)$, whereas $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_r}$ in [12]. In [12, p6], Dipper, Doty and Stoll defined operators E, S_i, \hat{S}_j acting on $V^{r,s}$. It is pointed in [13, Corollary 1.9] that e_1, g_i and g_j^* act on $V^{r,s}$ via E, S_{r-i} and \hat{S}_{r+j} , respectively. So, the condition for g_k acting on $V^{r,s}$ is the same as that for S_k in [12].

Theorem 4.6. [12, Theorem 1.4], [13, Theorem 6.1, Corollary 6.2] *Suppose $r, s \in \mathbb{Z}^{>0}$. Then $V^{r,s}$ is a $(\mathbf{U}_{\mathcal{Z}}, \mathcal{B}_{r,s})$ -bimodule. Moreover,*

- a) *there is an algebra epimorphism $\varphi : \mathbf{U}_{\mathcal{Z}} \twoheadrightarrow \text{End}_{\mathcal{B}_{r,s}}(V^{r,s})$;*
- b) *there is an algebra epimorphism $\psi : \mathcal{B}_{r,s} \twoheadrightarrow \text{End}_{\mathbf{U}_{\mathcal{Z}}}(V^{r,s})$. Further, ψ is an isomorphism if and only if $n \geq r+s$.*

In particular, Theorem 4.6 holds over an arbitrary field κ . The endomorphism algebra $\text{End}_{\mathcal{B}_{r,s}}(V^{r,s})$, which will be denoted by $S(n; r, s)$, is called the rational q -Schur algebra in [12].

In the remaining part of this section, unless otherwise stated, we assume $n \geq r+s$. We want to classify singular vectors of $V_\kappa^{r,s}$ with $n \geq r+s$ so as to establish explicit relationship between Weyl modules, indecomposable tilting modules of rational q -Schur algebras and cell modules, principal indecomposable modules of $\mathcal{B}_{r,s}$. This will give the required result on the decomposition numbers of $\mathcal{B}_{r,s,\kappa}$. If we allow $s = 0$, then $S(n; r, s)$ is known as q -Schur algebra $S(n, r)$ in [15].

In [17], Donkin has proved that rational Schur algebras in [11] are generalized Schur algebras. Note that generalized Schur algebras are always quasi-hereditary in the sense of [9]. So, rational Schur algebras are quasi-hereditary. The same is true for their quantizations. It is natural to modify his arguments to prove that rational q -Schur algebras are quasi-hereditary over κ . We need this fact when we classify singular vectors of $V^{r,s}$. Motivated by Dipper and Doty's work on quasi-heredity of rational Schur algebras in [11], we use arguments on cellular algebras to prove this fact.

For the simplification of notation, we use $\tilde{\lambda}$ instead of $\phi(f, \lambda)$ in the remaining part of this paper, where $(f, \lambda) \in \Lambda_{r,s}$ and $\phi(f, \lambda)$ is given in Lemma 4.3. Dipper et.al [12] defined column semistandard rational λ -tableaux $(\mathfrak{s}_1, \mathfrak{s}_2)$ in [12, 6.1]³. They proved that each $(\mathfrak{s}_1, \mathfrak{s}_2)$ corresponds to a unique column semistandard γ -tableau \mathbf{u} and vice versa, where $\gamma' = s\omega + \tilde{\lambda}'$. The transpose of this column semistandard γ -tableau is in fact the usual semistandard γ' -tableau. In the following, we denote γ by γ_λ to emphasis the (f, λ) in $\Lambda_{r,s}$.

Let $(\mathfrak{s}_1, \mathfrak{s}_2), (\mathfrak{t}_1, \mathfrak{t}_2)$ be two column semistandard rational λ -tableaux. Dipper etc [12] introduced rational bideterminants $((\mathfrak{s}_1, \mathfrak{s}_2) \mid (\mathfrak{t}_1, \mathfrak{t}_2)) \in A_q(n; r, s)$ where $A_q(n; r, s)$ is the linear dual of $S(n; r, s)$. It is proved in [12, Theorem 6.9] that the set of all bideterminants of column semistandard rational γ_λ -tableaux with $(f, \lambda) \in \Lambda_{r,s}$ forms a \mathcal{Z} -basis of $A_q(n; r, s)$.

Let $A_q(n, r + (n - 1)s)$ be the linear dual of $S(n, r + (n - 1)s)$ in [15]. For column semistandard λ' -tableau \mathbf{u}, \mathbf{v} with $\lambda' \in \Lambda^+(n, r + (n - 1)s)$, let $(\mathbf{u} \mid \mathbf{v}) \in A_q(n, r + (n - 1)s)$ be the corresponding bideterminant in [12]. In fact, it is the same as the bideterminant in [24], which is defined via the semistandard λ -tableaux $\mathfrak{s}, \mathfrak{t}$, the transposes of \mathbf{u}, \mathbf{v} , respectively.

It has been proved in [24] that the set of all bideterminants $(\mathbf{u} \mid \mathbf{v}) \in A_q(n, r + (n - 1)s)$ of column semistandard λ -tableaux $(\mathbf{u} \mid \mathbf{v})$ with $\lambda' \in \lambda(n, r + (n - 1)s)$ forms a \mathcal{Z} -basis of $A_q(n, r + (n - 1)s)$.

Dipper et.al [12, 6.5] have proved that $A_q(n; r, s)$ can be embedded into $A_q(n, r + (n - 1)s)$ via the linear map such that

$$\iota((\mathfrak{s}_1, \mathfrak{s}_2) \mid (\mathfrak{t}_1, \mathfrak{t}_2)) = q^c(\mathbf{u} \mid \mathbf{v}) \quad (4.3)$$

for some integer c . Here \mathbf{u}, \mathbf{v} are column semistandard γ_λ -tableaux with $(f, \lambda) \in \Lambda_{r,s}$, which correspond to column semistandard rational λ -tableaux $(\mathfrak{s}_1, \mathfrak{s}_2)$ and $(\mathfrak{t}_1, \mathfrak{t}_2)$, respectively. In this paper, we do not need the details about this.

Proposition 4.7. [12, Corollary 6.1] *Let π be the linear dual of ι . Then $\pi : S(n, r + (n - 1)s) \rightarrow S(n; r, s)$ is an algebra epimorphism over \mathcal{Z} .*

Let κ be a field which is a \mathcal{Z} -algebra. It is proved in [12] that the rational q -Schur algebra over κ is isomorphic to $S(n; r, s) \otimes_{\mathcal{Z}} \kappa$. For this reason, we identify $S_\kappa(n; r, s)$

³In fact, Dipper et.al called $(\mathfrak{s}_1, \mathfrak{s}_2)$ standard rational tableaux, which are different from usual standard tableaux in section 2.

with $S(n; r, s) \otimes_{\mathcal{Z}} \kappa$. The following result follows from certain results in [12]. The classical case has been given in [11] and [17]. As mentioned before, it can also follow from arguments similar to those in [17].

Theorem 4.8. *Suppose $n, r, s \in \mathbb{Z}^{>0}$ with $n \geq r + s$. Then $S_{\kappa}(n; r, s)$ is quasi-hereditary over κ in the sense of [9].*

Proof. Suppose $\alpha \in \Lambda^+(n, r + (n - 1)s)$. For usual semistandard α -tableaux \mathbf{u}, \mathbf{v} , let $Y_{\mathbf{u}, \mathbf{v}}^{\alpha} \in S_{\kappa}(n, r + (n - 1)s)$ be the codeterminant in [8, p 48].

We claim that $\pi(Y_{\mathbf{u}, \mathbf{v}}^{\alpha}) = 0$ if $\alpha \notin \Lambda$, where π is given in Proposition 4.7 and Λ is the ideal of $\Lambda^+(n, r + (n - 1)s)$ defined in Lemma 4.3.

In fact, if $\pi(Y_{\mathbf{u}, \mathbf{v}}^{\alpha}) \neq 0$, we can find a $(f, \lambda) \in \Lambda_{r, s}$ such that $\pi(Y_{\mathbf{u}, \mathbf{v}}^{\alpha})((\mathbf{s}_1, \mathbf{s}_2)|(\mathbf{t}_1, \mathbf{t}_2)) \neq 0$ for some rational bideterminant $(\mathbf{s}_1, \mathbf{s}_2)|(\mathbf{t}_1, \mathbf{t}_2)$ associated to a pair of column standard rational λ -tableaux $(\mathbf{s}_1, \mathbf{s}_2)$ and $(\mathbf{t}_1, \mathbf{t}_2)$. So, $Y_{\mathbf{u}, \mathbf{v}}^{\alpha}(\mathbf{u}_1 | \mathbf{v}_1) \neq 0$ where

$$q^c(\mathbf{u}_1 | \mathbf{v}_1) = \iota(\mathbf{s}_1, \mathbf{s}_2)|(\mathbf{t}_1, \mathbf{t}_2)$$

for some integer c (see (4.3)). Note that $(\mathbf{u}_1, \mathbf{v}_1)$ are a pair of semistandard γ'_{λ} -tableaux (or column semistandard γ_{λ} -tableaux if we use the notion of bideterminants in [12]). By [8, Theorem 12], we have $\gamma'_{\lambda} \supseteq \alpha$. Since Λ is an ideal of $\Lambda^+(n, r + (n - 1)s)$ and $\gamma'_{\lambda} \in \Lambda$, we have $\alpha \in \Lambda$, proving the claim. Counting the dimension of $S_{\kappa}(n; r, s)$, we see that the image of each codeterminant $Y_{\mathbf{u}, \mathbf{v}}^{\alpha}$ is nonzero in $S_{\kappa}(n; r, s)$ if $\alpha \in \Lambda$. Further, all non-zero of them form a basis of $S_{\kappa}(n; r, s)$ and the kernel of π is the κ -subspace generated by all $Y_{\mathbf{u}, \mathbf{v}}^{\alpha}$ with $\alpha \notin \Lambda$.

It is proved in [20, Theorem 5.5.1] that the codeterminant basis of a q-Schur algebra is a standard basis in the sense of [20, Definition 1.2.1]. One can check that the linear map τ sending any $Y_{\mathbf{u}, \mathbf{v}}^{\alpha}$ to $Y_{\mathbf{v}, \mathbf{u}}^{\alpha}$ is the required anti-involution. So, the codeterminant basis of a q-Schur algebra is a cellular basis in the sense of [22]. Therefore, $S_{\kappa}(n; r, s)$ is a cellular algebra with the cellular basis which consists of all the images of codeterminants $Y_{\mathbf{u}, \mathbf{v}}^{\alpha}$ for $\alpha \in \Lambda$. Further, non-isomorphic irreducible $S_{\kappa}(n; r, s)$ -modules are indexed by Λ . By [22, 3.10], $S_{\kappa}(n; r, s)$ is quasi-hereditary in the sense of [9]. \square

Definition 4.9. For each $(f, \lambda) \in \Lambda_{r, s}$ with $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, we write $\lambda' = (\alpha, \beta)$. We define $v_{0, \lambda} = v_{\mathbf{i}_{\lambda}} \otimes v_{\mathbf{j}_{\lambda}}^*$, and $v_{f, \lambda}$ for $f \geq 1$ as $v_{\mathbf{i}_{\lambda}} \otimes v^f \otimes v_{\mathbf{j}_{\lambda}}^* \in V^{r, s}$, where

- a) $\mathbf{i}_{\lambda} = (\alpha_{r-f}, \dots, 2, 1, \alpha_{r-f-1}, \dots, 2, 1, \dots, \alpha_1, \dots, 2, 1)$,
- b) $\mathbf{j}_{\lambda} = (n, n-1, \dots, n-\beta_1+1, \dots, n, n-1, \dots, n-\beta_{s-f}+1)$,
- c) $v^f = \sum_{k=1}^n v_k \otimes v^{f-1} \otimes v_k^*$ and $v^1 = \sum_{k=1}^n v_k \otimes v_k^*$.

Proposition 4.10. *Suppose $V_{\kappa}^{r, s}$ is defined over κ . For each $\mathbf{t} \in \mathcal{T}^{std}(\lambda')$ with $(f, \lambda) \in \Lambda_{r, s}$ and $d \in \mathcal{D}_{r, s}^f$, define $v_{\lambda, \mathbf{t}, d} = v_{f, \lambda} \mathbf{n}_{\lambda'} g_{d(\mathbf{t})} g_d \in V_{\kappa}^{r, s}$. Then $v_{\lambda, \mathbf{t}, d} \in V_{\kappa}^{r, s}$ is a singular vector with highest weight $\phi(f, \lambda)$.*

Proof. By Theorem 4.6, $V^{r, s}$ is a $(\mathbf{U}_{\mathcal{Z}}, \mathcal{B}_{r, s})$ -bimodule over \mathcal{Z} , where V is the free \mathcal{Z} -module with rank n . If we have $E_i(v_{f, \lambda} \mathbf{n}_{\lambda'}) = 0$ over $\mathbb{Q}(q)$, then $E_i^{\ell}/[\ell]!(v_{f, \lambda} \mathbf{n}_{\lambda'}) = 0$ over $\mathbb{Q}(q)$, forcing $E_i^{\ell}/[\ell]!(v_{f, \lambda} \mathbf{n}_{\lambda'}) = 0$ over \mathcal{Z} . By base change, $E_i^{\ell}/[\ell]!(v_{f, \lambda} \mathbf{n}_{\lambda'}) = 0$ over κ .

For any positive integer j with $j \leq f$, let

$$x_1 = 1^{\otimes(r-j)} \otimes E_i \otimes (K_i^{-1})^{\otimes(s+j-1)}, \text{ and } x_2 = 1^{\otimes(r+j-1)} \otimes E_i \otimes (K_i^{-1})^{\otimes(s-j)}.$$

If $v = v_1 \otimes v^j \otimes v_2 \in V^{r,s}$ where $v_1 \in V^{\otimes(r-j)}$ and v^j is given in Definition 4.9(c), by Lemma 4.1,

$$\begin{aligned} x_1 v &= q^{-1} v_1 \otimes v_i \otimes v^{j-1} \otimes v_{i+1}^* \otimes (K_i^{-1})^{\otimes(s-j)} v_2, \\ x_2 v &= -q^{-1} v_1 \otimes v_i \otimes v^{j-1} \otimes v_{i+1}^* \otimes (K_i^{-1})^{\otimes(s-j)} v_2. \end{aligned}$$

So, $(x_1 + x_2)v = 0$. Note that

$$\Delta^{r+s-1}(E_i) = \sum_{j=0}^{r+s-1} 1^{\otimes j} \otimes E_i \otimes (K_i^{-1})^{(r+s-j-1)}.$$

So, $E_i v_{f,\lambda}$ can be written as a linear combination of elements $v_{i_\lambda^k} \otimes v^f \otimes v_{j_\lambda}^*$ and $v_{i_\lambda} \otimes v^f \otimes v_{j_\lambda^k}^*$ where

- a) i_λ^k is obtained from i_λ by using i instead of $i+1$ in the sequence $(\alpha_k, \dots, 2, 1)$ if $i \leq \alpha_k - 1$,
- b) j_λ^k is obtained from j_λ by using i instead of $i-1$ in the sequence $(n, n-1, \dots, n-\beta_k+1)$ if $i \geq n-\beta_k+2$.

If i_λ^k is well defined, we write $w_k = (\sum_{j=1}^{k-1} \alpha_j + i, \sum_{j=1}^{k-1} \alpha_j + i + 1) \in \mathfrak{S}_{\lambda'}$. So,

$$v_{i_\lambda^k} \otimes v^f \otimes v_{j_\lambda}^* g_{w_k} \mathbf{n}_{\lambda'} = q^{-1} v_{i_\lambda^k} \otimes v^f \otimes v_{j_\lambda}^* \mathbf{n}_{\lambda'} = -q v_{i_\lambda^k} \otimes v^f \otimes v_{j_\lambda}^* \mathbf{n}_{\lambda'}, \text{ over } \mathcal{Z}.$$

This implies $v_{i_\lambda^k} \otimes v^f \otimes v_{j_\lambda}^* \mathbf{n}_{\lambda'} = 0$ over $\mathbb{Q}(q)$. Similarly, $v_{i_\lambda} \otimes v^f \otimes v_{j_\lambda^k}^* \mathbf{n}_{\lambda'} = 0$. So, $E_i(v_{f,\lambda} \mathbf{n}_{\lambda'}) = 0$ over $\mathbb{Q}(q)$. By Theorem 4.6, $E_i v_{\lambda,t,d} = (E_i(v_{f,\lambda} \mathbf{n}_{\lambda'})) g_{d(t)} g_d = 0$. Finally, it is easy to see that the weight of $v_{\lambda,t,d}$ is $\phi(f, \lambda)$. \square

Lemma 4.11. *Suppose $(f, \lambda) \in \Lambda_{r,s}$. Then $\{v_{\lambda,t,d} | t \in \mathcal{T}^{std}(\lambda'), d \in \mathcal{D}_{r,s}^f\}$ is κ -linearly independent.*

Proof. Let ξ_λ be obtained from $v_{f,\lambda}$ (see Definition 4.9) by using \mathbf{v} instead of v^f in Definition 4.9(c), where $\mathbf{v} = v_{\alpha_1+1} \otimes \dots \otimes v_{\alpha_1+f} \otimes v_{\alpha_1+f}^* \otimes \dots \otimes v_{\alpha_1+1}^*$ and $\lambda' = (\alpha, \beta)$.

If $\sum_{t,d} a_{t,d} v_{\lambda,t,d} = 0$, $a_{t,d} \in \kappa$, by Proposition 4.4, $\sum_t a_{t,d} \xi_\lambda \mathbf{n}_{\lambda'} g_{d(t)} g_d = 0$ for any fixed d . Since g_d is invertible, $\sum_t a_{t,d} \xi_\lambda \mathbf{n}_{\lambda'} g_{d(t)} = 0$.

It is well known that $V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n,r)} \mathbf{m}_\lambda \mathcal{H}_r$ as right \mathcal{H}_r -modules, where \mathbf{m}_λ is obtained from that in section 2 by using q^{-1} instead of q . The corresponding isomorphism sends $v_{i_\lambda,d}$ to $q^{-l(d)} \mathbf{m}_\lambda g_d^4$ for any distinguished right coset representative d of $\mathfrak{S}_\lambda / \mathfrak{S}_r$. In particular, if we consider $V_\kappa^{r-f} \otimes \mathbf{v} \otimes W_\kappa^{s-f}$ as right $\mathcal{H}_{r-f} \otimes \mathcal{H}_{s-f}$ -module, then $\xi_\lambda \mathbf{n}_{\lambda'} g_{d(t)}$ corresponds to $\mathbf{m}_\lambda g_{w_\lambda} \mathbf{n}_{\lambda'} g_{d(t)}$ up to a non-zero scalar in κ , where $w_\lambda = d(t_\lambda)$. Note that $\mathcal{H}_{r-f} \otimes \mathcal{H}_{s-f}$ are generated by g_{f+i} and g_{f+j}^* for all positive integers i, j with $f+i \leq r-1$ and $f+j \leq s-1$. By [14, Theorem 5.6], $a_{t,d} = 0$, for all possible t and d . \square

⁴If $(g_i - q)(g_i + q^{-1}) = 0$, then the corresponding isomorphism sends $v_{i_\lambda,d}$ to $q^{l(d)} \mathbf{m}_\lambda g_d$.

By Proposition 4.10, each \mathbf{U}_κ -module generated by $v_{\lambda,t,d}$ is a highest weight module with highest weight $\tilde{\lambda} := \phi(f, \lambda)$ in Lemma 4.3. By the universal property of Weyl modules in [1, 1.20], $\mathbf{U}_\kappa v_{\lambda,t,d}$ is a quotient of $\Delta(\tilde{\lambda})$ where $\Delta(\tilde{\lambda})$ is the Weyl module of \mathbf{U}_κ with respect to the highest weight $\tilde{\lambda}$. We will use this fact in Proposition 4.12.

Proposition 4.12. *If $(f, \lambda) \in \Lambda_{r,s}$, then there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{U}_\kappa}(\Delta(\tilde{\lambda}), V_\kappa^{r,s}) \cong C(f, \lambda')$$

as right $\mathcal{B}_{r,s,\kappa}$ -modules if $n \geq r + s$.

Proof. For each $(d, \mathbf{t}) \in \mathcal{D}_{r,s}^f \times \mathcal{T}^{std}(\lambda')$, let $M_{d,\mathbf{t}} = \mathbf{U}_\kappa v_{\lambda,t,d} \subset V^{r,s}$. By Proposition 4.10, $v_{\lambda,t,d} \in V^{r,s}$ is a highest weight vector with highest weight $\tilde{\lambda}$. So, there is a unique \mathbf{U}_κ -epimorphism (up to a scalar) from $\Delta(\tilde{\lambda})$ to $M_{d,\mathbf{t}}$ sending highest weight vector to highest weight vector. Such a \mathbf{U}_κ -homomorphism will be denoted by $f_{\lambda,t,d}$. Since $M_{d,\mathbf{t}}$ is a submodule of $V^{r,s}$, $f_{\lambda,t,d}$ results in a homomorphism in $\mathrm{Hom}_{\mathbf{U}_\kappa}(\Delta(\tilde{\lambda}), V_\kappa^{r,s})$. By abuse of notation, we denote this homomorphism by $f_{\lambda,t,d}$. By Lemma 4.11, $\{f_{\lambda,t,d} \mid (d, \mathbf{t}) \in \mathcal{D}_{r,s}^f \times \mathcal{T}^{std}(\lambda')\}$ is κ -linear independent.

Now, we compute the dimension of $\mathrm{Hom}_{\mathbf{U}_\kappa}(\Delta(\tilde{\lambda}), V_\kappa^{r,s})$. It is known that $V = \Delta(\varepsilon_1) = \nabla(\varepsilon_1)$ and $V^* = \Delta(-\varepsilon_n) = \nabla(-\varepsilon_n)$, where $\nabla(\varepsilon_1)$ is the co-Weyl module with highest weight ε_1 . So, both V and V^* are tilting \mathbf{U}_κ -module. It is known that the tensor product of tilting module is again a tilting module [16]. So, $V_\kappa^{r,s}$ is a tilting module for \mathbf{U}_κ .

Over \mathcal{Z} , we have the similar functor $\diamond = \mathrm{Hom}_{\mathbf{U}_\mathcal{Z}}(-, V^{r,s})$. In this case,

$$\Delta(\tilde{\lambda})^\diamond = \mathrm{Hom}_{\mathbf{U}_\mathcal{Z}}(\Delta(\tilde{\lambda}), V^{r,s})_{\mathbb{Q}(q)} \cap \mathrm{Hom}_\mathcal{Z}(\Delta(\tilde{\lambda}), V^{r,s}).$$

By Corollary C.19 in [10], $\Delta(\tilde{\lambda})^\diamond$ is \mathcal{Z} -projective. So, the localization of $\Delta(\tilde{\lambda})^\diamond$ has constant rank for any prime ideal \mathfrak{p} of \mathcal{Z} (See §7.7 in [25]). Therefore, the dimension of $\Delta(\tilde{\lambda})^\diamond$ over any κ is equal to that over $\mathbb{Q}(q)$. Note that $V^{r,s}$ is complete reducible as left $\mathbf{U}_{\mathbb{Q}(q)}$ -module and the multiplicity of $\Delta(\tilde{\lambda})$ in $V_{\mathbb{Q}(q)}^{r,s}$ is $\dim C(f, \lambda')$ [27, Lemma 6.5], which is the cardinality of $\{f_{\lambda,t,d} \mid (d, \mathbf{t}) \in \mathcal{D}_{r,s}^f \times \mathcal{T}^{std}(\lambda')\}$. Therefore, $\{f_{\lambda,t,d} \mid (d, \mathbf{t}) \in \mathcal{D}_{r,s}^f \times \mathcal{T}^{std}(\lambda')\}$ is a κ -basis of $\mathrm{Hom}_{\mathbf{U}_\kappa}(\Delta(\tilde{\lambda}), V_\kappa^{r,s})$.

By definition, $\mathrm{Hom}_{\mathbf{U}_\kappa}(\Delta(\tilde{\lambda}), V_\kappa^{r,s})$ is the right $\mathcal{B}_{r,s,\kappa}$ -module such that

$$f.h(x) = f(x) \cdot h, \forall h \in \mathcal{B}_{r,s,\kappa}, f \in \mathrm{Hom}_{\mathbf{U}_\kappa}(\Delta(\tilde{\lambda}), V_\kappa^{r,s}) \text{ and } x \in \Delta(\tilde{\lambda}).$$

Finally, one can check easily that the linear isomorphism which sends $f_{\lambda,t,d}$ to $e^f \mathbf{m}_\lambda g_{w_\lambda} \mathbf{n}_{\lambda'} g_{d(\mathbf{t})} g_d + \mathcal{B}_{r,s,\kappa}^f$ is a right $\mathcal{B}_{r,s,\kappa}$ -homomorphism. Now, the result follows from Lemma 2.6. \square

The following result gives a classification of singular vectors in $V_\kappa^{r,s}$ with highest weight $\tilde{\lambda} = \phi(f, \lambda)$ for all $(f, \lambda) \in \Lambda_{r,s}$.

Theorem 4.13. *For any $(f, \lambda) \in \Lambda_{r,s}$ with $0 \leq f \leq \min\{r, s\}$, let $S(\lambda) = \{v_{\lambda,t,d} \mid \mathbf{t} \in \mathcal{T}^{std}(\lambda'), d \in \mathcal{D}_{r,s}^f\}$. Then $S(\lambda)$ is a basis of the κ -space spanned by all singular vectors of $V_\kappa^{r,s}$ with highest weight $\tilde{\lambda}$.*

Proof. Note that the weight of each singular vector $v \in V^{r,s}$ is of form $\tilde{\lambda}$, for some $(f, \lambda) \in \Lambda_{r,s}$, let M_v be the left \mathbf{U}_κ -module generated by v . Then there is a homomorphism ϕ_v from Weyl module $\Delta(\tilde{\lambda})$ to M sending highest weight vector to highest weight vector. By Proposition 4.12, ϕ_v can be written as a κ -linear combination of $f_{\lambda,t,d}$'s. Applying these homomorphisms to v_λ , we see that v can be written as a κ -linear combination of $v_{\lambda,t,d}$'s, proving the result. \square

5. DECOMPOSITION NUMBERS OF $\mathcal{B}_{r,s,\kappa}$ WITH $\rho^2 \in q^{2\mathbb{Z}}$

In this section, we establish explicit relationship between decomposition numbers of $\mathcal{B}_{r,s,\kappa}$ and those for (rational) q -Schur algebras. First, we discuss the case when $e < \infty$. Therefore, $\rho = q^{n+2ke}$ for any $k \in \mathbb{Z}$. In this case, we can always assume that n is big enough. We remark that we use Dipper, Doty and Stoll's presentation for $\mathcal{B}_{r,s}$ in [13]. So, we have to use q^{-1}, ρ^{-1} instead of q, ρ respectively, if we use our results in section 2.

Definition 5.1. Suppose $r, s, n \in \mathbb{Z}^{>0}$. Let $\diamond = \text{Hom}_{S_\kappa(n;r,s)}(-, V_\kappa^{r,s})$ and let $\clubsuit = \text{Hom}_{\mathcal{B}_{r,s,\kappa}}(-, V_\kappa^{r,s})$, where V_κ is the κ -vector space with $\dim_\kappa V_\kappa = n$.

For each left $S_\kappa(n;r,s)$ -module M , let $M^\diamond = \text{Hom}_{S_\kappa(n;r,s)}(M, V_\kappa^{r,s})$. Then M^\diamond is a right $\mathcal{B}_{r,s,\kappa}$ -module. Similarly, for each right $\mathcal{B}_{r,s,\kappa}$ -module N , let $N^\clubsuit = \text{Hom}_{\mathcal{B}_{r,s,\kappa}}(N, V_\kappa^{r,s})$. Then N^\clubsuit is a left $S_\kappa(n;r,s)$ -module.

The following result, which follows from Proposition 4.12, is the key part of our method for determining the decomposition numbers of $\mathcal{B}_{r,s}$ when $\rho^2 \in q^{2\mathbb{Z}}$ and q is a root of unity. Via it, we can set up explicit relationship between indecomposable direct summands of V^{rs} and principal indecomposable $\mathcal{B}_{r,s}$ -modules.

Proposition 5.2. Suppose $n \geq r + s$. If $\tilde{\lambda}$ is a highest weight of $V_\kappa^{r,s}$, then $\Delta(\tilde{\lambda})^\diamond \cong C(f, \lambda')$ as right $\mathcal{B}_{r,s,\kappa}$ -modules.

Proof. We consider $V^{r,s}$ over $\mathbb{Q}(q)$ with generic q . It is complete reducible as left $\mathbf{U}_{\mathbb{Q}(q)}$ -module. Therefore, it can be decomposed into direct summand of irreducible $\mathbf{U}_{\mathbb{Q}(q)}$ -modules, say L^ν 's with highest weight ν 's. By Lemma 4.3, each ν is of form $\tilde{\lambda}$ for some $(f, \lambda) \in \Lambda_{r,s}$. Further, by Theorem 4.13, L^λ can be generated by certain $v_{\lambda,t,d}$ for some $t \in \mathcal{T}^{std}(\lambda')$ and $d \in D_{r,s}^f$. Note that $\mathbf{U}_{\mathbb{Q}(q)} v_{\lambda,t,d}$ has \mathcal{Z} -form $\mathbf{U}_{\mathcal{Z}v_{\lambda,t,d}}$. Therefore, $\mathbf{U}_\kappa v_{\lambda,t,d}$, which is a left \mathbf{U}_κ -submodule of $V_\kappa^{r,s}$, can be identified with the Weyl module $\Delta(\tilde{\lambda})$ of \mathbf{U}_κ . So, $\Delta(\tilde{\lambda})$ can be considered as a left $S_\kappa(n;r,s)$ -module. Now, the result follows from Proposition 4.12. \square

Lemma 5.3. For $r, s, n \in \mathbb{Z}^{>0}$, let $\mathcal{B}_{r,s,\kappa}$ and $S_\kappa(n;r,s)$ be defined over κ with $\rho = q^n$.

- a) $\mathcal{B}_{r,s,\kappa}^\clubsuit \cong V_\kappa^{r,s}$ as left $S_\kappa(n;r,s)$ -modules.
- b) If $n \geq r + s$, then $(V_\kappa^{r,s})^\diamond \cong \mathcal{B}_{r,s,\kappa}$ as right $\mathcal{B}_{r,s,\kappa}$ -modules.

Proof. (a) is trivial and (b) follows from Theorem 4.6(b). \square

In the remaining part of this section, we keep the assumption that $n \geq r + s$. Recall that a tilting module for quantum group is a module with Weyl filtration and co-Weyl filtration. Similarly, we have the notion of tilting modules for quasi-hereditary algebras. See, e.g. [16].

Recall that an indecomposable tilting module is called a **partial tilting module**. By Theorem 1 in [16, p208], $V_\kappa^{r,s}$ is a direct sum of certain partial tilting modules of $S_\kappa(n; r, s)$. Note that any dominant weight of $V_\kappa^{r,s}$ is of form $\tilde{\lambda}$ for some $(f, \lambda) \in \Lambda_{r,s}$. So, any partial tilting module which is a direct summand of $V_\kappa^{r,s}$ is of form $T(\tilde{\lambda})$ with highest weight $\tilde{\lambda}$ for some $(f, \lambda) \in \Lambda_{r,s}$. Let $(T(\tilde{\lambda}) : \Delta(\tilde{\mu}))$ be the multiplicity of $\Delta(\tilde{\mu})$ in $T(\tilde{\lambda})$. It is well known that $(T(\tilde{\lambda}) : \Delta(\tilde{\mu}))$ is independent of a Weyl filtration of $T(\tilde{\lambda})$. We are going to use $(T(\tilde{\lambda}) : \Delta(\tilde{\mu}))$'s to determine decomposition numbers of $\mathcal{B}_{r,s,\kappa}$ over the field κ .

Let $S_\kappa(n; r, s)\text{-mod}$ (resp. $\mathcal{B}_{r,s,\kappa}\text{-mod}$) be the category of left $S_\kappa(n; r, s)$ -modules (resp. $\mathcal{B}_{r,s,\kappa}$ -modules). For each left $S_\kappa(n; r, s)$ -module M , $\text{Hom}_{S_\kappa(n; r, s)}(V_\kappa^{r,s}, M)$ is a left $\mathcal{B}_{r,s,\kappa}$ -module such that, for any $x \in V_\kappa^{r,s}$, $b \in \mathcal{B}_{r,s,\kappa}$ and $\phi \in \text{Hom}_{S_\kappa(n; r, s)}(V_\kappa^{r,s}, M)$,

$$(b\phi)(x) = \phi(xb). \quad (5.1)$$

Also, $V_\kappa^{r,s} \otimes_{\mathcal{B}_{r,s,\kappa}} N$ is a left $S_\kappa(n; r, s)$ -module for any left $\mathcal{B}_{r,s,\kappa}$ -module N .

Definition 5.4. Let \mathbf{f} and \mathbf{g} be two functors

$$\begin{aligned} \mathbf{f} : S_\kappa(n; r, s)\text{-mod} &\longrightarrow \mathcal{B}_{r,s,\kappa}\text{-mod} \\ M &\longmapsto \text{Hom}_{S_\kappa(n; r, s)}(V_\kappa^{r,s}, M) \\ \mathbf{g} : \mathcal{B}_{r,s,\kappa}\text{-mod} &\longrightarrow S_\kappa(n; r, s)\text{-mod} \\ N &\longmapsto V_\kappa^{r,s} \otimes_{\mathcal{B}_{r,s,\kappa}} N \end{aligned}$$

Since \mathbf{f} and \mathbf{g} are adjoint pairs (see e.g. [31, Theorem 2.11]), we have a κ -linear isomorphism

$$\text{Hom}_{S_\kappa(n; r, s)}(\mathbf{g}(N), M) \cong \text{Hom}_{\mathcal{B}_{r,s,\kappa}}(N, \mathbf{f}(M)), \quad (5.2)$$

for any left $S_\kappa(n; r, s)$ -module M and any left $\mathcal{B}_{r,s,\kappa}$ -module N .

Lemma 5.5. [12, Theorem 6.11] $S_\kappa(n; r, s) = \varphi(\mathbf{U}'_\kappa)$, where φ is given in Theorem 4.6(a) and $\mathbf{U}'_\kappa = \mathbf{U}_\kappa(\mathfrak{sl}_n)$.

For any $i, 1 \leq i \leq n$, let $v_i = v_n \otimes \cdots \otimes v_{i+1} \otimes v_{i-1} \otimes \cdots \otimes v_1 \in V_\kappa^{\otimes n-1}$, where

$$\hat{i} = (1, 2, \dots, i-1, i+1, \dots, n). \quad (5.3)$$

The following result has been given in [12, Lemma 2.2].

Lemma 5.6. There is a well defined \mathbf{U}'_κ -monomorphism $\varphi : V_\kappa^* \longrightarrow V_\kappa^{\otimes n-1}$ such that $\varphi(v_i^*) = (-q)^i v_i \mathbf{n}_{(n-1)}$ where $\mathbf{n}_{(n-1)} = \sum_{w \in \mathfrak{S}_{n-1}} (-q)^{\ell(w)} T_w$.

There is a \mathbf{U}'_κ -monomorphism from $V_\kappa^{r,s}$ to $V_\kappa^{\otimes r+(n-1)s}$ induced by φ in Lemma 5.6. By abuse of notation, we denote this monomorphism by φ . Recall that there is an anti-automorphism τ of \mathbf{U}_κ given by

$$\tau(q^{h_i}) = q^{h_i}, \tau(E_i) = F_i, \tau(F_i) = E_i.$$

For any positive integer m , Stokke [34] defined a symmetric bilinear form $(\ , \) : V_\kappa^{\otimes m} \times V_\kappa^{\otimes m} \longrightarrow \kappa$ such that

$$(v_{\mathbf{i}}, v_{\mathbf{j}}) = q^{\beta(\mathbf{i})} \delta_{\mathbf{i}, \mathbf{j}}, \quad (5.4)$$

where $\mathbf{i}, \mathbf{j} \in I(n, m)$, $\beta(\mathbf{i})$ is the number of the pairs (a, b) for which $a < b$ and $i_a \neq i_b$. It is proved in [34, Theorem 5.2] that the bilinear form $(\ , \)$ in (5.4) is the \mathbf{U}_κ -contravariant form in the sense

$$(uv, w) = (v, \tau(u)w), u \in \mathbf{U}_\kappa, v, w \in V_\kappa^{\otimes m}. \quad (5.5)$$

The following can be considered as a counterpart of the form in (5.4).

Definition 5.7. Let $(\ , \) : V_\kappa^{r,s} \times V_\kappa^{r,s} \longrightarrow \kappa$ be the bilinear form such that, for any $(\mathbf{i}, \mathbf{j}), (\mathbf{k}, \mathbf{l}) \in I(n, r) \times I^*(n, s)$,

$$(v_{\mathbf{i}|\mathbf{j}}, v_{\mathbf{k}|\mathbf{l}}) = q^{2(j_1+j_2+\dots+j_s)+\beta(\widehat{\mathbf{i}|\mathbf{j}})} \delta_{\mathbf{i}, \mathbf{k}} \delta_{\mathbf{j}, \mathbf{l}}, \quad (5.6)$$

where $\widehat{\mathbf{i}|\mathbf{j}} = (i_1, i_2, \dots, i_r, \hat{j}_1, \hat{j}_2, \dots, \hat{j}_s)$ and \hat{j}_i is defined in (5.3)

Lemma 5.8. *Let ϕ be the bilinear form on $V_\kappa^{r,s}$ defined in (5.6).*

- a) ϕ is non-degenerate symmetric and \mathbf{U}'_κ -contravariant.
- b) $\phi(xb, y) = \phi(x, y\sigma(b))$, for all $b \in \mathcal{B}_{r,s,\kappa}$ and $x, y \in V_\kappa^{r,s}$, where σ is the anti-involution defined in Lemma 2.2.

Proof. In fact, the bilinear form ϕ is \mathbf{U}' -contravariant over \mathcal{Z} and hence over κ . In order to see it, we consider the \mathbf{U}' -contravariant form on $V^{\otimes r+(n-1)s}$ in (5.4) over $\mathbb{Q}(q)$. By Lemma 5.6, there is a \mathbf{U}' -monomorphism $\varphi : V^{r,s} \longrightarrow V^{\otimes r+(n-1)s}$ over $\mathbb{Q}(q)$. So, there is a \mathbf{U}' -contravariant form, say $(\ , \)_q$ on $V^{r,s}$, such that

$$(x, y)_q = (\varphi(x), \varphi(y)), x, y \in V^{r,s}.$$

By (5.4), $(\ , \)_q = (\sum_{w \in \mathfrak{S}_{n-1}} q^{2\ell(w)})^s \phi$, forcing ϕ to be \mathbf{U}' -contravariant on $V^{r,s}$ over $\mathbb{Q}(q)$ and hence over \mathcal{Z} . The others in (a) are clear.

We claim $\phi(v_{\mathbf{i}|\mathbf{j}}T_i, v_{\mathbf{k}|\mathbf{l}}) = \phi(v_{\mathbf{i}|\mathbf{j}}, v_{\mathbf{k}|\mathbf{l}}T_i)$. Without loss of generality, we can assume that $\mathbf{j} = \mathbf{l}$, $i = 1$, and $\mathbf{i} = (i_1, i_2)$, $i_1 < i_2$ and $i_1 = k_2$, $i_2 = k_1$. In this case, we have $\beta(\widehat{\mathbf{k}|\mathbf{l}}) = \beta(\widehat{\mathbf{i}|\mathbf{j}})$. A routine computation verifies our claim. By symmetry, $\phi(v_{\mathbf{i}|\mathbf{j}}T_j^*, v_{\mathbf{k}|\mathbf{l}}) = \phi(v_{\mathbf{i}|\mathbf{j}}, v_{\mathbf{k}|\mathbf{l}}T_j^*)$. Finally, we verify $\phi(v_{\mathbf{i}|\mathbf{j}}e_1, v_{\mathbf{k}|\mathbf{l}}) = \phi(v_{\mathbf{i}|\mathbf{j}}, v_{\mathbf{k}|\mathbf{l}}e_1)$. In this case, we can assume that $\mathbf{i} = (i)$, $\mathbf{k} = (k)$, and $j_1 = i$, $l_1 = k$, $j_m = l_m$, for $m = 2, 3, \dots, s$. So, $\beta(\widehat{\mathbf{k}|\mathbf{l}}) = \beta(\widehat{\mathbf{i}|\mathbf{j}})$ and

$$\phi(v_{\mathbf{i}|\mathbf{j}}e_1, v_{\mathbf{k}|\mathbf{l}}) = q^{2(i+k)-n-1} q^{\beta(\widehat{\mathbf{k}|\mathbf{l}})} = q^{2(i+k)-n-1} q^{\beta(\widehat{\mathbf{i}|\mathbf{j}})} = \phi(v_{\mathbf{i}|\mathbf{j}}, v_{\mathbf{k}|\mathbf{l}}e_1).$$

This completes the proof of (b). □

By Lemma 5.5, any left $S_\kappa(n; r, s)$ -module can be considered as a left \mathbf{U}'_κ -module. For any left $S_\kappa(n; r, s)$ -module N , let N° be the left $S_\kappa(n; r, s)$ -module such that $N^\circ = N^*$ as κ -vector space, and the action is given by

$$(u\phi)(x) = \phi(\tau(u)x), x \in N, u \in \mathbf{U}'_\kappa, \phi \in N^* \quad (5.7)$$

For any right $\mathcal{B}_{r,s,\kappa}$ -module M , let M° be the right $\mathcal{B}_{r,s,\kappa}$ -module such that $M^\circ = M^*$ as κ -vector space, and the action is given by

$$(\phi b)(y) = \phi(y\sigma(b)), y \in M, b \in \mathcal{B}_{r,s,\kappa}, \phi \in M^* \quad (5.8)$$

Lemma 5.9. *As $(S_\kappa(n; r, s), \mathcal{B}_{r,s,\kappa})$ bi-modules, $V_\kappa^{r,s} \cong (V_\kappa^{r,s})^\circ$.*

Proof. The required isomorphism Φ follows from Lemma 5.8 if we define $\Phi : V_\kappa^{r,s} \longrightarrow (V_\kappa^{r,s})^\circ$ such that $\Phi(x)(y) = (x, y)$ for all $x, y \in V_\kappa^{r,s}$, where $(\ , \)$ is given in Definition 5.7. \square

Lemma 5.10. *Suppose that T is an indecomposable direct summand of $S_\kappa(n; r, s)$ -module $V_\kappa^{r,s}$. Then $\mathbf{gf}(T) \cong T$.*

Proof. Since we are assuming that $r + s \leq n$, $\mathbf{f}(V_\kappa^{r,s}) \cong \mathcal{B}_{r,s,\kappa}$ and $\mathbf{gf}(V_\kappa^{r,s}) \cong V_\kappa^{r,s}$. It is easy to see that there is an epimorphism from $gf(M)$ to M for any indecomposable direct summand M of $V_\kappa^{r,s}$ as $S_\kappa(n; r, s)$ -modules. Comparing the dimensions yields the isomorphism as required. \square

Lemma 5.11. *Any partial tilting module which appears as an indecomposable direct summand of $V_\kappa^{r,s}$ is of form $T(\lambda)$ for some $(f, \lambda') \in \Lambda_{r,s}$ with λ being e -regular. Further we have the following isomorphisms as left $\mathcal{B}_{r,s,\kappa}$ -modules:*

- a) for $(\ell, \mu) \in \Lambda_{r,s}$, $\mathbf{f}(\nabla(\tilde{\mu})) \cong \text{Hom}_{S_\kappa(n;r,s)}(\Delta(\tilde{\mu}), V_\kappa^{r,s});$
- b) $\mathbf{f}(T(\tilde{\lambda})) \cong P(f, \lambda').$

Proof. It follows from Lemma 5.9 that $V_\kappa^{r,s} \cong (V_\kappa^{r,s})^\circ$. By [16, Proposition 4.1.6], $\nabla(\tilde{\mu}) \cong \Delta(\tilde{\mu})^\circ$. In order to prove (a), it suffices to prove the following isomorphism as left $\mathcal{B}_{r,s,\kappa}$ -modules:

$$\text{Hom}_{S_\kappa(n;r,s)}(\Delta(\tilde{\mu}), V_\kappa^{r,s}) \cong \text{Hom}_{S_\kappa(n;r,s)}((V_\kappa^{r,s})^\circ, \Delta(\tilde{\mu})^\circ). \quad (5.9)$$

Obviously, $\Psi : \text{Hom}_{S_\kappa(n;r,s)}(\Delta(\tilde{\mu}), V_\kappa^{r,s}) \rightarrow \text{Hom}_{S_\kappa(n;r,s)}((V_\kappa^{r,s})^\circ, \Delta(\tilde{\mu})^\circ)$ given by

$$\Psi(\phi)(v^*) : x \mapsto v^*(\phi(x)),$$

for any $\phi \in \text{Hom}_{S_\kappa(n;r,s)}(\Delta(\tilde{\mu}), V_\kappa^{r,s})$, $v \in V_\kappa^{r,s}$, $x \in \Delta(\tilde{\mu})$ is a κ -linear isomorphism. For any $b \in \mathcal{B}_{r,s,\kappa}$, we have $b\Psi(\phi)(v^*) = \Psi(\phi)(v^*b)$ and

$$\begin{aligned} \Psi(b\phi)(v^*) : x &\mapsto v^*((b\phi)(x)) = v^*(\phi(x)\sigma(b)), \\ \Psi(\phi)(v^*b) : x &\mapsto v^*b(\phi(x)) = v^*(\phi(x)\sigma(b)). \end{aligned}$$

So $\Psi(b\phi) = b\Psi(\phi)$, and (a) follows.

(b) By [3, II, Proposition 2.1(c)], The functor \mathbf{f} induces a category equivalence between the direct sums of direct summands of the $S_\kappa(n; r, s)$ -module $V_\kappa^{r,s}$ and the projective $\mathcal{B}_{r,s,\kappa}$ -modules. So $\mathbf{f}(T(\tilde{\mu}))$ is an indecomposable projective $\mathcal{B}_{r,s,\kappa}$ -module. For any $(k, \nu') \in \Lambda_{r,s}$ by Lemma 5.10, Proposition 5.2, (5.2) and (a), we have κ -linear isomorphism

$$\begin{aligned} \text{Hom}_{S_\kappa(n;r,s)}(T(\tilde{\mu}), \nabla(\tilde{\nu})) &\cong \text{Hom}_{S_\kappa(n;r,s)}(\mathbf{gf}(T(\tilde{\mu})), \nabla(\tilde{\nu})) \\ &\cong \text{Hom}_{S_\kappa(n;r,s)}(\mathbf{f}(T(\tilde{\mu})), \mathbf{f}(\nabla(\tilde{\nu}))) \cong \text{Hom}_{\mathcal{B}_{r,s,\kappa}}(P(f, \lambda'), C(k, \nu')) \end{aligned} \quad (5.10)$$

for some $(f, \lambda') \in \Lambda_{r,s}$ such that $\mathbf{f}(T(\tilde{\mu})) = P(f, \lambda')$ with λ being e -regular. We remark that $C(k, \nu')$ is considered as a right $\mathcal{B}_{r,s,\kappa}$ -module in Proposition 5.2. Using anti-involution σ in Lemma 2.2, it can be considered as the left $\mathcal{B}_{r,s,\kappa}$ -module in (5.10). By [30, Lemma 2.18],

$$\dim_{\kappa} \operatorname{Hom}_{\mathcal{B}_{r,s}}(P(f, \lambda'), C(k, \nu')) = [C(k, \nu') : D^{f, \lambda'}]. \quad (5.11)$$

We have $(\ell, \mu') \supseteq (f, \lambda')$ by assuming $\mu = \nu$. If $\nu = \lambda$, then

$$\operatorname{Hom}_{S_{\kappa}(n;r,s)}(T(\tilde{\mu}), \nabla(\tilde{\lambda})) \neq 0,$$

forcing $\tilde{\lambda} \trianglelefteq \tilde{\mu}$. So, $(\ell, \mu') \trianglelefteq (f, \lambda')$, $f = \ell$ and $\mu = \lambda$. This proves (b). \square

Theorem 5.12. *Suppose $(f, \lambda'), (\ell, \mu') \in \Lambda_{r,s}$ such that λ is e -regular. Then*

$$(T(\tilde{\lambda}) : \Delta(\tilde{\mu})) = [C(\ell, \mu') : D^{f, \lambda'}].$$

Proof. Since $(T(\tilde{\lambda}) : \Delta(\tilde{\mu})) = \dim_{\kappa} \operatorname{Hom}_{S_{\kappa}(n;r,s)}(T(\tilde{\mu}), \nabla(\tilde{\nu}))$, the result follows from (5.10)–(5.11) \square

In the remaining part of this section, we consider right $\mathcal{B}_{r,s,\kappa}$ -modules. Of course, Theorem 5.12 can be read for right $\mathcal{B}_{r,s,\kappa}$ -modules since any left $\mathcal{B}_{r,s,\kappa}$ -module can be considered as a right $\mathcal{B}_{r,s,\kappa}$ -module via the anti-involution σ in Lemma 2.2.

Theorem 5.13. *Suppose $e < \infty$. Let $\mathcal{B}_{r,s,\kappa}$ be defined over κ with defining parameter $\rho = q^n$. If $(f, \lambda'), (\ell, \mu') \in \Lambda_{r,s}$, then $C(f, \lambda')$ and $C(\ell, \mu')$ are in the same $\mathcal{B}_{r,s,\kappa}$ block if and only if $\Delta(\tilde{\lambda})$ and $\Delta(\tilde{\mu})$ are in the same $S_{\kappa}(n; r, s)$ -block.*

Proof. Suppose that $C(f, \lambda')$ and $C(\ell, \mu')$ are in the same $\mathcal{B}_{r,s,\kappa}$ -block. Without loss of any generality, we assume that $[C(f, \lambda') : D^{(\ell, \mu')}] \neq 0$. So, μ is e -regular. By Theorem 5.12, $(T(\tilde{\mu}) : \Delta(\tilde{\lambda})) \neq 0$. Since $\Delta(\tilde{\mu})$ is the unique bottom section of any Weyl filtration of the partial tilting module $T(\tilde{\mu})$, $\Delta(\tilde{\lambda})$ and $\Delta(\tilde{\mu})$ are in the same $S_{\kappa}(n; r, s)$ -block.

Conversely, let Y be a $\mathcal{B}_{r,s,\kappa}$ -module which is an indecomposable direct summand of $V_{\kappa}^{r,s}$. Then $Y^{\clubsuit} \neq 0$. By definition of $S_{\kappa}(n; r, s)$, Y^{\clubsuit} is a direct summand of $S_{\kappa}(n; r, s)$.

We claim that Y^{\clubsuit} is indecomposable. Otherwise, Y^{\clubsuit} is a direct sum of certain principal indecomposable $S_{\kappa}(n; r, s)$ -modules, say $P(\tilde{\lambda})$'s. By Proposition 4.12, $0 \neq P(\tilde{\lambda})^{\diamond}$, which is a direct summand of $\mathcal{B}_{r,s,\kappa}$ -module $V_{\kappa}^{r,s}$. Counting the number of indecomposable direct summands of $\mathcal{B}_{r,s,\kappa}$ -module $V_{\kappa}^{r,s}$ gives a contradiction. So, both Y^{\clubsuit} and $P(\tilde{\lambda})^{\diamond}$ are indecomposable and hence $Y^{\clubsuit} = P(\tilde{\lambda})$ for some $\tilde{\lambda} \in \Lambda^+(r, s)$.

Suppose $\Delta(\tilde{\lambda})$ and $\Delta(\tilde{\mu})$ are in the same $S_{\kappa}(n; r, s)$ -block. Without loss of any generality, we can assume $(P(\tilde{\lambda}) : \Delta(\tilde{\mu})) \neq 0$. Applying \diamond to $P(\tilde{\lambda})$, we see that both $C(f, \lambda')$ and $C(\ell, \mu')$ appear as sections of a cell filtration of the indecomposable right $\mathcal{B}_{r,s,\kappa}$ -module $P(\tilde{\lambda})^{\diamond}$. So, $C(f, \lambda')$ and $C(\ell, \mu')$ are in the same block. \square

By Theorem 5.13, we know that blocks of $\mathcal{B}_{r,s,\kappa}$ can be determined by those of rational q -Schur algebras. We remark that we will study blocks of $\mathcal{B}_{r,s,\kappa}$ in details elsewhere.

Each indecomposable direct summand of right $\mathcal{B}_{r,s,\kappa}$ -module $V_\kappa^{r,s}$ will be called a Young module. Let $Y(f, \lambda') = P(\tilde{\lambda})^\diamond$. Using standard arguments on tilting module $V_\kappa^{r,s}$, we have the following result immediately.

Corollary 5.14. *Suppose $e < \infty$. Let $\mathcal{B}_{r,s,\kappa}$ be defined over κ with $\rho = q^n$ for $n \gg 0$. Suppose $(f, \lambda'), (\ell, \mu') \in \Lambda_{r,s}$. Then $Y(f, \lambda')$ has a filtration of right cell modules of $\mathcal{B}_{r,s,\kappa}$ with bottom section $C(f, \lambda')$. Further, the multiplicity of $C(\ell, \mu')$ in the previous filtration of $Y(f, \lambda')$ is $(P(\tilde{\lambda}) : \Delta(\tilde{\mu})) = [\Delta(\tilde{\mu}) : L^{\tilde{\lambda}}]$, where $L^{\tilde{\lambda}}$ is the irreducible $S_\kappa(n; r, s)$ -module with highest weight $\tilde{\lambda}$.*

The following result is motivated by Donkin and Tange's work in [18].

Theorem 5.15. *If $\rho^2 = q^{2a}$ for some $a \in \mathbb{Z}$ with $|a| \leq r + s - 2$ and $e = \infty$, then $[C(f, \lambda') : D^{\ell, \mu'}]$ is the same as that under the assumption $e \gg 0$.*

Proof. Let t be an indeterminate. We consider $\kappa[t, t^{-1}]$ which is the localization of $\kappa[t]$ at t . So, $\kappa[t, t^{-1}]$ is a Dedekind ring with quotient field $K = \kappa(t)$. Since $\mathcal{B}_{r,s}$ is cellular over $\mathbb{Z}[q, q^{-1}, \rho, \rho^{-1}, (q - q^{-1})^{-1}]$, we see that $\mathcal{B}_{r,s}$ is free over $\kappa[t, t^{-1}]$ with defining parameters ρ and t such that $\rho^2 = t^{2a}$. Further, it is a $\kappa[t, t^{-1}]$ -lattice of $\mathcal{B}_{r,s,K}$ in the sense of [18, 5.2]. Let ε^2 be primitive k -th root of unity in κ and let $M_\varepsilon \subset \kappa[t, t^{-1}]$ be the maximal ideal generated by $t - \varepsilon$. Then $\kappa = \kappa[t, t^{-1}]/M_\varepsilon$. In this case, we use κ_ε instead of κ so as to emphasis ε .

Let $Grot(\mathcal{B}_{r,s,F})$ be the Grothendieck group of finite dimension $\mathcal{B}_{r,s,F}$ -modules over F , a field which is a $\kappa[t, t^{-1}]$ -algebra. By [18, 5.2(1)], $Grot(\mathcal{B}_{r,s,K}) \cong Grot(\mathcal{B}_{r,s,\kappa_\varepsilon})$ if k , the order of ε^2 , is big enough. Since $\mathcal{B}_{r,s}$ is a cellular algebra, any cell module $C(f, \lambda)$ of $\mathcal{B}_{r,s}$ can be considered as $\kappa[t, t^{-1}]$ -lattice of the corresponding cell module $C(f, \lambda)_K$ of $\mathcal{B}_{r,s,K}$. Therefore, the decomposition matrices of $\mathcal{B}_{r,s}$ over K and κ_ε are the same if the order of ε^2 is big enough. \square

Finally, we explain why decomposition numbers of $\mathcal{B}_{r,s,\kappa}$ can be computed via those for q -Schur algebras if $\rho^2 \in q^{2\mathbb{Z}}$.

Suppose $(f, \lambda') \in \Lambda_{r,s}$. We have \mathbf{U}_κ -module $\Delta(\tilde{\lambda})$. When λ is e -regular, we have $T(\tilde{\lambda})$, an indecomposable direct summand of $V_\kappa^{r,s}$ with respect to the highest weight $\tilde{\lambda}$. Both $T(\tilde{\lambda})$ and $\Delta(\tilde{\lambda})$ are rational representations of \mathbf{U}_κ . When we consider the restriction of such modules to $\mathbf{U}_\kappa(\mathfrak{sl}_n)$, they are isomorphic to $T(\tilde{\lambda} + s\omega), \Delta(\tilde{\lambda} + s\omega)$ with $\omega = (1, \dots, 1) \in \Lambda^+(n)$, the corresponding polynomial representations of \mathbf{U}_κ . Further, we have the following well-known equalities:

$$(T(\tilde{\lambda}) : \Delta(\tilde{\mu})) = (T(\tilde{\lambda} + s\omega) : \Delta(\tilde{\mu} + s\omega)) = [\Delta(\alpha) : L^\beta] \quad (5.12)$$

where α (resp. β) is the conjugate of $\tilde{\mu} + s\omega$ (resp. $\tilde{\lambda} + s\omega$). We remark that the last equality follows from [16, Proposition 4.1e]. So, the decomposition numbers for $\mathcal{B}_{r,s,\kappa}$ can be computed via those for q -Schur algebras if $q^2 \in q^{2\mathbb{Z}}$. Finally, if $\rho^2 \notin q^{2\mathbb{Z}}$, by Theorems 3.2, decomposition numbers of $\mathcal{B}_{r,s,\kappa}$ can be computed by those for Hecke algebras associated to symmetric groups.

If the ground field κ is \mathbb{C} , we can use Ariki's result in [2] and Varagnolo and Vasserot's results in [35]. In the latter case, we have to use q instead of q^{-1} . In [35], Varagnolo and Vasserot used $U_q(\mathfrak{gl}_n)$ in [26]. By [1, Remark 1.25], we need to use $w_0(s\omega + \tilde{\lambda})$ instead of our $s\omega + \tilde{\lambda}$ when we use corresponding result in [35], where w_0 is the longest element in \mathfrak{S}_n . In summary, when the ground field is \mathbb{C} , decomposition numbers of $\mathcal{B}_{r,s,\kappa}$ can be computed via the values of inverse Kazhdan-Lusztig polynomials at $q = 1$ associated to certain extended affine Weyl groups of type A . We leave the details to the reader.

Cox and De Visscher [6] proved that decomposition numbers of walled Brauer algebras over \mathbb{C} are either 0 or 1. This should correspond to our result for $\mathcal{B}_{r,s}$ over \mathbb{C} with $o(q) = \infty$ and $\rho^2 \in q^{2\mathbb{Z}}$. Finally, it is natural to ask whether one can find results for quantum general linear superalgebras and quantized walled Brauer algebras similar to those for general linear Lie superalgebras and walled Brauer algebras in [5].

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H.R. DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, 200062, CHINA

E-mail address: hbrui@math.ecnu.edu.cn

L.S. DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, 200062, CHINA

E-mail address: 52110601013@student.ecnu.edu.cn